Uncertainty and asset markets in GET

Lecture 9

November 22, 2015

Uncertainty

- GET is capable of incorporating uncertainty albeit a little awkwardly.
- We focus on a pure exchange economy.
- The way to do this is to introduce states of the world $S = \{s_1, ..., s_R\}$.
- Consumer h evaluates that state s_i comes about with probability π_{hi}.
- The basic innovation is to regard, say, a loaf of bread in two different states as two different commodities.
- This means that consumption bundles and allocations are indexed by the states and become really long.

Uncertainty

Definition

A state contingent commodity vector/bundle $x = (x_{11}, ..., x_{L1}, ..., x_{1R}, ..., x_{LR}) \in \mathbb{R}_+^{LR}$ provides a consumer bundle $(x_{1s}, ..., x_{Ls}) \in \mathbb{R}_+^L$ in state $s \in S$.

- Notice that above is a description of a contingent bundle for a consumer.
- An allocation, a bundle for each consumer, is an element of $\mathbb{R}_+^{LRH}$.
- Consumer h has endowment $\omega_h = (\omega_{11h}, ..., \omega_{L1h}, ..., \omega_{1Rh}, ..., \omega_{LRh})$.
- Preferences are defined over contingent commodity bundles.

Uncertainty

- The consumer has a Bernoulli-utility function for each state and evaluates bundles by weighing the states by their probabilities so that $x_h \succeq_h x'_h$ if and only if

$$\sum_{s \in S} \pi_{sh} u_{sh} \left(x_{1sh}, \ldots, x_{Lsh}\right) \geq \sum_{s \in S} \pi_{sh} u_{sh} \left(x'_{1sh}, \ldots, x'_{Lsh}\right)$$

- Formally, the economy with states and state contingent commodities is equivalent to the standard version, and all the results we have there remain true.

- In particular, Walras-equilibria exist and they are Pareto-efficient.

- When the Bernoulli-utility functions are concave we immediately see that in equilibrium there is efficient risk sharing.

Uncertainty

Information

- Think of information becoming more and more accurate with time, and to model this assume that there are dates $t \in \{0, 1, ..., T\}$.
- A partition of a set A is a collection of sets $\{A_i\}_{i=1}^n$ such that $A_i \cap A_j = \emptyset$ when $i \neq j$, and $\cup_{i=1}^n A_i = A$.
- When $A = S$ the subsets of S, $S_i = A_i$, are called events, and a partion $\mathscr{L}$ of S is called an information structure.
- When $s, s' \in S_i$ then an agent regards it as possible that the state of the world is s as well as s'.
- Information revelation is modelled by a sequence of information structures $(\mathscr{L}_0, \mathscr{L}_1, ..., \mathscr{L}_T)$ such that if $Q \in \mathscr{L}_t$ then there exists $R \in \mathscr{L}_{t-1}$ such that $Q \subseteq R$.

Uncertainty

- Each consumer could have his/her own information structure but to keep notation from exploding we assume that they have a common sequence of information structures.

- A pair (t, E) where $E \in \mathscr{L}_t$ is called a date-event.

- There are L commodities and now we denote contingent commodities by index lt meaning that commodity l is available at date t.

- We still need to index the commodities by states so x_{lts} is the amount of commodity l available at date t in state s (along with other states that belong to the same event).

Uncertainty

- We can define the double-indexed commodities, of which there are $L(T+1)$, as our new commodities, and then everything is as in the standard setting.

- The only thing we have to take care is that bundles and allocations are measurable, i.e., if $s, s' \in S_i$ then $x_{lts} = x_{lts'}$.

- This way the above temporal structure can be made atemporal, and the economy is formally like the standard setting.

- The equilibrium of this economy is called the Arrow-Debreu equilibrium.

Uncertainty

Definition

Arrow-Debreu equilibrium consists of allocation
$x^* = (x_1^*, x_2^*, ..., x_H^*) \in \mathbb{R}_+^{LRH}$,
$x_h^* = (x_{11h}^*, ..., x_{L1h}^*, ..., x_{1Rh}^* ..., x_{LRh}^*) \in \mathbb{R}_+^{LR}$, and a system of prices
$p = (p_{11}, ..., p_{L1}, ..., p_{1R}, ..., p_{LR}) \in \mathbb{R}_+^{LR}$ such that
i) for every h x_h^* is the maximal element in the budget set
$\left\{x_h \in \mathbb{R}_+^{LR} : px_h \leq p\omega_h\right\}$ and
ii) markets clear $\sum_{h=1}^{H} x_h^* = \sum_{h=1}^{H} \omega_h$.

Uncertainty

- In the Arrow-Debreu economy there are markets for all contingent commodities, and trading happens before any of the uncertainty is resolved.
- Assume that at $t = 0$ the agents can trade just one commodity but then at the succeeding dates there are spotmarkets at each state.
- Now only S forward markets exist (one for each state) in the one commodity that is traded at $t = 0$.
- To proceed we need to postulate that the agents have (correct) expectations about the spot prices at time $t = 1$.
- The expected price at state s is denoted by $p_s = (p_{1s}, ..., p_{Ls})$ and the expected vector of prices is $p = (p_{s_1}, ..., p_{s_R})$.
- At time $t = 0$ there is contingent trade in good 1, and its prices are given by $q = (q_{s_1}, ..., q_{s_R})$.

Uncertainty

- Given the prices consumer h makes a consumption plan for contingent commodities $(z_{s_1 h}, \ldots, z_{s_R h}) \in \mathbb{R}^R$ and for spot markets $(x_{s_1 h}, \ldots, x_{s_R h})$ where one has to be careful with notation as $x_{sh} = (x_{1sh}, \ldots, x_{Lsh}) \in \mathbb{R}^L_+$.
- The consumer's problem is given by

$$\max_{\substack{(z_{s_1 h}, \ldots, z_{s_R h}) \in \mathbb{R}^R \\ (x_{s_1 h}, \ldots, x_{s_R h}) \in \mathbb{R}^{LR}_+}} U_h(x_{s_1 h}, \ldots, x_{s_R h})$$

$$s.t. \quad \begin{array}{c} \sum_{s \in S} q_s z_{sh} \leq 0 \\ p_s x_{sh} \leq p_s \omega_{sh} + p_{1s} z_{sh} \end{array} \quad \forall s \in S$$

- The first budget constraint does not restrict much; it is well possible that, say, $z_{s_1 h} < -\omega_{1 s_1 h}$.
- In this case the consumer sells short and has to acquire sufficient amount of good 1 in the spot market to honour his/her commitments.
- The requirement of positive consumption (i.e., wealth) at each state guarantees that things do not get out of control.

Uncertainty

Definition

A Radner-equilibrium consists of prices at time $t = 0$ for contingent first good commodities $q = (q_{s_1}, ..., q_{s_R})$ and spot prices $p = (p_{s_1}, ..., p_{s_R})$ at time $t = 1$, for each consumer h consumption plan $\left(z^*_{s_1 h}, ..., z^*_{s_R h}\right)$ at time $t = 0$, and consumption plan $\left(x^*_{s_1 h}, ..., x^*_{s_R h}\right)$ at time $t = 1$ such that the consumption plans solve the above maximisation problem, and $\sum_{h \in \mathscr{H}} z^*_{sh} \leq 0$ and $\sum_{h \in \mathscr{H}} x^*_{sh} \leq \sum_{h \in \mathscr{H}} \omega_{sh}$ for each $s \in S$.

Uncertainty

Theorem

i) Assume that an Arrow-Debreu equilibrium is given by allocation $x^* \in \mathbb{R}_+^{LRH}$ *and a contingent commodity prices* $p = (p_{s_1}, ..., p_{s_R}) \in \mathbb{R}_{++}^{LR}$. *There are prices* $q = (q_{s_1}, ..., q_{s_R}) \in \mathbb{R}_{++}^{R}$ *for the contingent commodity-1 and consumption plans* $z^* = (z_1^*, ..., z_H^*) \in \mathbb{R}^{RH}$ *such that the consumption plans* x^*, z^* *and prices* q *and* p *constitute a Radner-equilibrium.*
ii) If prices $q = (q_{s_1}, ..., q_{s_R}) \in \mathbb{R}_{++}^{R}$ *for the contingent commodity-1 and spot prices* $p = (p_{s_1}, ..., p_{s_R}) \in \mathbb{R}_{++}^{LR}$, *and consumption plans* $z^* = (z_1^*, ..., z_H^*) \in \mathbb{R}^{RH}$ *and* $x^* \in \mathbb{R}_+^{LRH}$ *constitute a Radner-equilibrium then there are multipliers* $(\mu_{s_1}, ..., \mu_{s_R}) \in \mathbb{R}_{++}^{R}$ *such that allocation* x^* *and the contingent commodities prices* $(\mu_{s_1} p_{s_1}, ..., \mu_{s_R} p_{s_R}) \in \mathbb{R}_{++}^{R}$ *constitute an Arrow-Debreu equilibrium.*

Uncertainty

Proof.

i) A good guess is to make $q_s = p_{1s}$ for each $s \in S$. Let us next compare the budget sets of consumer h under Arrow-Debreu prices and Radner-prices. The Arrow-Debreu budget set is $B_h^{AD} = \left\{ (x_{s_1 h}, ..., x_{s_R h}) \in \mathbb{R}_+^{LR} : \sum_{s \in S} p_s (x_{sh} - \omega_{sh}) \leq 0 \right\}$ while the Radner budget set is

$$B_h^R =$$

$$\Bigg\{ (x_{s_1 h}, ..., x_{s_R h}) \in \mathbb{R}_+^{LR} : \exists (z_{s_1 h}, ..., z_{s_R h}) \in \mathbb{R}^R \, s.t.$$

$$\sum_{s \in S} q_s z_{sh} \leq 0 \, and \, p_s (x_{sh} - \omega_{sh}) \leq p_{1s} z_{sh}, \forall s \in S \Bigg\}$$

Assume that $x_h \in B_h^{AD}$, and let $z_{sh} = \frac{1}{p_{1s}} p_s (x_{sh} - \omega_{sh})$. Now $\sum_{s \in S} q_s z_{sh} = \sum_{s \in S} p_s (x_{sh} - \omega_{sh}) \leq 0$ and $p_s (x_{sh} - \omega_{sh}) = p_{1s} z_{sh}$ for all $s \in S$. This shows that $B_h^{AD} \subseteq B_h^R$. □

Uncertainty

Proof.

Assume that $x_h \in B_h^R$. This means that there exists $(z_{s_1 h}, \ldots, z_{s_R h})$ such that $\sum_{s \in S} q_s z_{sh} \leq 0$ and $p_s (x_{sh} - \omega_{sh}) \leq p_{1s} z_{sh}$ for all $s \in S$. Summing the latter inequality over the states yields $\sum_{s \in S} p_s (x_{sh} - \omega_{sh}) \leq \sum_{s \in S} p_{1s} z_{sh} = \sum_{s \in S} q_s z_{sh} \leq 0$. This shows that $B_h^R \subseteq B_h^{AD}$. □

Uncertainty

Proof.

ii) Choose $\mu_s = \frac{q_s}{p_{1s}}$ for all $s \in S$, and write the Radner budget set as

$$B_h^R =$$

$$\Big\{(x_{s_1h}, ..., x_{s_Rh}) \in \mathbb{R}_+^{LR} : \exists (z_{s_1h}, ..., z_{s_Rh}) \in \mathbb{R}^R \, s.t.$$

$$\sum_{s \in S} q_s z_{sh} \leq 0 \, and \, \mu_s p_s (x_{sh} - \omega_{sh}) \leq p_{1s} z_{sh}, \forall s \in S \Big\}$$

But then one can mimic the proof of i) and write the budget set in the Arrow-Debreu form. QED □

Asset markets

- Assets pay money or real goods conditional on states.
- Let us assume that asset payments are in good 1.
- There are two dates $t = 0$ and $t = 1$.
- Let there be S states like in 19.E in MWG.
- Asset is characterised by a return vector $r = (r_1, \dots r_S) \in \mathbb{R}^S$.
- r_s is what the asset pays in state s.

Asset markets

- Options constitute an important class of assets.
- Underlying an option is another asset; assume its return vector is given by $r \in \mathbb{R}^S$.
- A call option is characterised by a strike price c.
- A unit of an option gives the holder a right to buy a unit of the underlying asset at price c after the uncertainty has resolved.
- The return vector of the asset is $r(c) = (max\{0, r_1 - c\}, max\{0, r_2 - c\}, ..., max\{0, r_S - c\})$.
- At state s an option holder exercises his/her right if $r_s - c > 0$.

Asset markets

- Assume that there is a given set of assets called asset structure tradable at $t = 0$.
- Assume there are K assets.
- The price of assets at $t = 0$ is given by $q = (q_1, ..., q_K) \in \mathbb{R}^K$.
- The trades are denoted by $z = (z_1, ..., z_K) \in \mathbb{R}^K$, and called a portfolio.

Definition

Asset prices $q = (q_1, ..., q_K) \in \mathbb{R}^K$ and spot prices $p_s = (p_{1s}, ..., p_{Ls}) \in \mathbb{R}^L$ for all s, and for all consumers i a portfolio $z_i^* = (z_{1i}^*, ..., z_{Ki}^*) \in \mathbb{R}^K$, and consumption plan $x_i^* = (x_{1i}^*, ..., x_{Si}^*) \in \mathbb{R}^{LS}$ is a Radner-equilibrium if z_i^* and x_i^* solve

$$max_{z_i^* \in \mathbb{R}^K x_i^* \in \mathbb{R}^{LS}} U_i(x_{1i}, ..., x_{Si})$$

$$s.t. \sum_k q_k z_k \leq 0$$

$$p_s x_{si} \leq p_s \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk} \forall s$$

$$\sum_i z_{ki}^* \leq 0$$

$$\sum_i x_{si}^* \leq \sum_i \omega_{si} \forall s, k$$

Asset markets

- Note that $p_{1s} = 1$ is a possible normalisation; this is done in the sequel.
- Things are clearer in matrix-notation and we define the $S \times K$ return matrix

$$R = \begin{matrix} r_{11} & . & . & . & . & r_{1K} \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ r_{S1} & . & . & . & . & r_{SK} \end{matrix}$$

where row s keeps track of returns of different assets in state s, and column k keeps track of asset k's returns in different states.

Asset markets

- Now the budget constraint becomes

$$B_i(p,q,R) = \left\{ x \in \mathbb{R}_+^{LS} : z_i \in \mathbb{R}^K, qz_i \leq 0 \text{ and } \begin{array}{c} p_1\left(x_{1i} - \omega_{1i}\right) \\ \cdot \\ \cdot \\ \cdot \\ p_S\left(x_{Si} - \omega_{Si}\right) \end{array} \leq Rz_i \right\}$$

Theorem

Assume that $0 \neq r_k \geq 0$ for all k. Then for every vector of asset prices $q \in \mathbb{R}^K$ in a Radner-equilibrium can be found multipliers $\mu = (\mu_1, ..., \mu_S) \geq 0$ such that $q_k = \sum_s \mu_s r_{sk}$, or $q^T = \mu R$.

Proof.

Asset price vector $q \in \mathbb{R}^K$ is arbitrage free if there is no portfolio $z = (z_1, ..., z_K)$ such that $qz \leq 0$, $0 \neq Rz \geq 0$. It is straightforward that in equilibrium q must be arbitrage free. We first show that if $q \in \mathbb{R}^K$ is arbitrage free then there exists a vector of multipliers $\mu = (\mu_1, ..., \mu_S) \geq 0$ such that $q^T = \mu R$. It is clear that $q_k > 0$ for all k. We can also assume that each row of R has strictly positive elements. □

Proof.

Consider the set

$$V = \left\{ v \in \mathbb{R}^S : v = Rz, z \in \mathbb{R}^K \text{ and } qz = 0 \right\}$$

Since q is arbitrage free $V \cap \mathbb{R}^S_+ \setminus \{0\} = \emptyset$. Both are convex and there exists a separating hyperplane; there exists $\mu' = (\mu'_1, ..., \mu'_S) \geq 0$ such that $\mu' v \leq 0$ for $v \in V$ and $\mu' w \geq 0$ for $w \in \mathbb{R}^S_+$. Since $-v \in V$ whenever $v \in V$ it must be the case that $\mu' v = 0$ for all $v \in V$.

We claim that q^T is proportional to $\mu' R \in \mathbb{R}^K$. First $0 \neq \mu' R \geq 0^T$. If the claim does not hold there exists $\bar{z} \in \mathbb{R}^K$ such that $q\bar{z} = 0$ and $\mu' R\bar{z} > 0$. But if $v = R\bar{z}$ then $v \in V$ and $\mu' v \neq 0$ which is a contradiction. Consequently, $q^T = \alpha \mu' R$ for some $\alpha > 0$; let $\mu = \alpha \mu'$.

As equilibrium asset prices must be arbitrage free the result follows □

Asset markets

Definition

Asset structure with $S \times K$ return matrix is complete if *rank* $R = S$.

- With complete asset structure transfer of wealth over all states is possible.
- If $S = 4$ and a primary asset has returns $r = (4, 3, 2, 1)$ we can have a complete asset structure with options $r(3.5)$, $r(2.5)$ and $r(1.5)$.

Theorem

If the asset structure is complete the equilibrium is Pareto-efficient.

- With assets the important thing is the range of the return matrix
$$range\ R = \left\{ v \in \mathbb{R}^S : v = Rz, z \in \mathbb{R}^K \right\}$$
- It tells the possible wealth vectors that are achievable with a given asset structure.

Theorem

Let there be two assets structures with return matrices R and R'. If range $R =$ range R' then the consumption plans x^*in Radner equilibria corresponding to R and R' are equal.

Exercise 19.E.4.Suppose that $r_3 = \alpha_1 r_1 + \alpha_2 r_2$. Show that in equilibrium $q_3 = \alpha_1 q_1 + \alpha_2 q_2$. Assume first that $q_3 > \alpha_1 q_1 + \alpha_2 q_2$. Then portfolio $z = (\alpha_1 q_3, \alpha_2 q_3, -\alpha_1 q_1 - \alpha_2 q_2)$ must return zero or $qz = 0$. Now
$\sum_k p_{1s} r_{sk} z_k = p_{1s}(r_{s1}\alpha_1 q_3 + r_{s2}\alpha_2 q_3 - r_{s3}(\alpha_1 q_1 + \alpha_2 q_2)) =$
$p_{1s} r_{s3}(q_3 - (\alpha_1 q_1 + \alpha_2 q_2))$ for each s. Since $0 \neq r_3 \geq 0$ we have $\sum_k p_{1s} r_{sk} z_k \geq 0$ for each s and at least one strict inequality. Consumers can always increase their wealth, and utility, by adding z to their portfolio. But this cannot happen in equilibrium. If $q_3 < \alpha_1 q_1 + \alpha_2 q_2$ then we can show analogously that consumers can always profitably subtract z from their portfolio. Thus, it must be the case that $q_3 = \alpha_1 q_1 + \alpha_2 q_2$.