Core convergence

Lecture 8

November 17, 2015

Core convergence

- Whatever one thinks about the relevance of general equilibrium theory it has a surprisingly strong foundation provided by cooperative game theory.
- Consider an exchange economy $\mathscr{E} = ((\succeq_h, \omega_h)_{h \in \mathscr{H}})$ where $\succeq_h$ is a continuous, strictly convex and strictly monotone preference defined on $\mathbb{R}^L_+$.
- We say that $x = (x_1, ..., x_H) \in X = X_1 \times ... \times X_H$ is an allocation if $\sum_{h \in \mathscr{H}} x_h = \sum_{h \in \mathscr{H}} \omega_h$.
- An allocation x belongs to the core, $x \in C(\mathscr{E})$, if there does not exist a coalition $S \subseteq \mathscr{H}$ and an allocation y such that $\sum_{s \in S} y_s = \sum_{s \in S} \omega_s$ and $y_s \succ_s x_s$ for all $s \in S$.
- We say that no coalition improves upon x.
- Notice that assuming strict preference for all agents is w.l.o.g. as we have assumed that the preferences are strictly/strongly monotonic.

- One immediately sees that core allocations are individually rational and Pareto-optimal.
- Otherwise single-agent coalitions or the grand coalition could block an allocation.
- The relation between the core and an exchange economy can be established by showing that an exchange economy can be regarded as a balanced game.
- Balanced games have a non-empty core.
- We skip this part and take a shortcut by utilising the fact that a Walrasian equilibrium exists.

- Denote the set of Walrasian equilibrium allocations by $W(\mathscr{E})$.

Theorem

$W(\mathscr{E}) \subseteq C(\mathscr{E})$.

Proof.

Assume that there exists a Walrasian equilibrium allocation $x \notin C(\mathscr{E})$, and let the corresponding equilibrium price be p. Now there must be a coalition $S \subseteq \mathscr{H}$ and an allocation y such that $\sum_{s \in S} y_s = \sum_{s \in S} \omega_s$ and $y_s \succ_s x_s$ for all $s \in S$. As x is a Walrasian equilibrium allocation it must be the case that $py_s > p\omega_s$ for all $s \in S$. Summing over S we get $\sum_{s \in S} py_s > \sum_{s \in S} p\omega_s$ which is equivalent to $p \sum_{s \in S} y_s > p \sum_{s \in S} \omega_s$ which contradicts $\sum_{s \in S} y_s = \sum_{s \in S} \omega_s$. QED □

- Notice that this is another way of proving the First Welfare Theorem.
- We also know that the core is non-empty because the Walrasian equilibrium exists.
- We are going to increase the size of the economy in a specific way which defines so called r-replica economies.

Definition

Let A be a finite set and $n \in \mathbb{Z}_+$. Denote by nA a multiset which consists of n identical copies of each member of A. Denote $\mathscr{H}_1 \equiv 1\mathscr{H} = \mathscr{H}$, and let $\mathscr{H}_r = r\mathscr{H}$ for $r \in \mathbb{Z}_+$. Let $\mathscr{E}_1 \equiv \mathscr{E} = ((\succeq_h, \omega_h)_{h \in \mathscr{H}})$ and $\mathscr{E}_r = ((\succeq_h, \omega_h)_{h \in \mathscr{H}_r})$.

Theorem

Assume the the agents have strictly convex preferences. Then any allocation in $C(\mathscr{E}_r)$, $r \in \mathbb{Z}_+$, *has equal treatment property, or* $x_{hi} = x_{hj}$ *for* $h \in \mathscr{H}$, $i,j \in \{1,2,...,r\}$, $x \in C(\mathscr{E}_r)$.

Proof.

Consider feasible allocation $(x_{11},\ldots,x_{1r},x_{21},\ldots,x_{2r},\ldots,x_{H1},\ldots,x_{Hr})$ meaning $\sum_{h,i} x_{hi} = r\sum_h \omega_h$. Assume that the consumers are ordered such that $x_{h1} \preceq_h x_{hj}$ for $h \in \mathscr{H}$, $j \in \{1,2,...,r\}$. Consider coalition $S = \{11, 21, 31, ..., H1\}$ and allocation where type h gets $\frac{1}{r}\sum_{i=1}^{r} x_{hi}$. Because the original allocation is feasible so it this. If the original allocation does not have equal treatment property for some type h, then this allocation is a strict improvement for h and all other types do at least as well as in the original allocation. QED □

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Theorem

(Debreu and Scarf) Assume that $\sum_h \omega_h >> 0$ *and preferences* $\succeq_h$ *are strictly increasing and strictly convex. If equal treatment allocation* x *belongs to the core of* $\mathscr{E}_r$ *for each* $r \in \mathbb{Z}_+$ *then it is a Walrasian equilibrium allocation, or* $\cap_{r=1}^{\infty} C(\mathscr{E}_r) \subseteq W(\mathscr{E})$, *for some strictly positive price vector.*

Proof.

Consider the set of net trades that makes agent of type h better-off than s/he is at x_h

$$Z_h(x_h, \omega_h) = \left\{ \zeta \in \mathbb{R}^L : \zeta + \omega_h \succ_h x_h,\ \zeta + \omega_h \geq 0 \right\}$$

The aim is to show that $Z_h(x_h, \omega_h)$ lies in the positive side of a hyperplane defined by the candidate equilibrium price p, which is still to be introduced; and this must hold for all agents. □

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Proof.

To this end define

$$Z^*(x,\omega) = co\left(\cup_{h\in\mathscr{H}} Z_h(x_h,\omega_h)\right)$$

where co indicates the convex hull. Assume that the intersection of $Z^*(x,\omega)$ and the strict negative orthant of $\mathbb{R}^L$ is non-empty. Then there exists $0 >> \zeta = \sum_h \lambda_h \zeta_h$ where $\zeta_h \in Z_h(x_h,\omega_h)$ and $\lambda_h \geq 0$ and $\sum_h \lambda_h = 1$. Strict negativity and continuity of sum and multiplication allows us to find $\zeta = \sum_h \lambda_h \zeta_h$ such that λ_h are rational numbers. □

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Proof.

If we make these rationals to have the same denominator we have the expression

$$\zeta = \sum_h \frac{M_h}{M} \zeta_h$$

Consider next economy $\mathscr{E}_r$ where $r \geq M$ and a coalition which has M_h members of type $h \in \mathscr{H}$. The coalition can generate net trades ζ_h for each of its members $h \in \mathscr{H}$. Since $\sum_h \frac{M_h}{M} \zeta_h \leq 0$ also $\sum_h M_h \zeta_h \leq 0$ meaning that the net trades are feasible. This results each member of the coalition to be strictly better than at allocation x; consequently x cannot belong to the core for $\mathscr{E}_r$, $r \geq M$. □

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Proof.

Thus, if the x is in the core for all $\mathscr{E}_r$ then $Z^*(x,\omega)$ must be disjoint from the strict negative orthant of $\mathbb{R}^L$. As both of these are convex sets there must be a non-zero $p \in \mathbb{R}^L$ that separates the sets, or there must be a scalar such that $p \cdot \zeta \geq \beta$ for all $\zeta \in Z^*(x,\omega)$ and $p \cdot \zeta \leq \beta$ for all strictly negative ζ. But then $p \geq 0$. Also $\beta \geq 0$ because sequence $(\zeta_n)_{n\in\mathbb{N}}$ can be chosen from the negative orthant such that $p\zeta_n$ goes to zero. □

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Proof.

Next we show that $p \cdot (x_h - \omega_h) \geq \beta$. Take non-negative $z \in \mathbb{R}^L$ and $\varepsilon > 0$ and consider bundle $x_h + \varepsilon z$. Because preferences are strictly increasing this in strictly preferred to x_h. This means that $x_h + \varepsilon z - \omega_h \in Z_h(x_h, \omega_h)$ and $p \cdot (x_h + \varepsilon z - \omega_h) \geq \beta$. Letting ε go to zero we get $p \cdot (x_h - \omega_h) \geq \beta$.

Then we show that $\beta = 0$ and $p \cdot (x_h - \omega_h) = 0$. Since x is feasible $\sum_h x_h \leq \sum_h \omega_h$. Taking the inner product of each side with p we get $\sum_h p(x_h - \omega_h) \leq 0$. But $p \cdot (x_h - \omega_h) \geq \beta$ which shows the claim. □

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Proof.

We still have to show that if a bundle is strictly preferred to x_h it costs strictly more. We show this by first showing that p is strictly positive. Assuming $L = 1$ this is clear; otherwise we would not have a non-zero price given by the separating hyperplane theorem. Assume then that $L > 1$ and that price of good i is zero, or $p_i = 0$. By SHPT there must be j such that $p_j > 0$. There is a strictly positive amount of good j in the economy, and some consumer h must get a strictly positive amount of it. Add a unit of good i to x_h and make h strictly better-off. □

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Proof.

Now we can take a little of good j away from h and s/he is still better-off than at x_h. Call this new bundle, $\widetilde{x}$, and notice that $\widetilde{x} - \omega_h \in Z_h(x_h, \omega_h)$ so that $p(\widetilde{x} - \omega_h) \geq 0$. But since $p(x_h - \omega_h) = 0$ and the difference between x_h and $\widetilde{x}$ is that the latter has a unit of more of good i and a little less of good j $p(\widetilde{x} - \omega_h) \geq 0$ cannot hold; the price of good i was zero. Thus, p is strictly positive. □

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Proof.

Finally, assume that $\hat{x}$ is strictly preferred to x_h. For $\lambda < 1$ sufficiently close to unity also $\lambda \hat{x}$ is strictly preferred to x_h or $\lambda \hat{x} - \omega_h \in Z_h(x_h, \omega_h)$ and $p\lambda \hat{x} \geq p\omega_h$. Clearly, $p\hat{x} > 0$ and $p\hat{x} > \lambda p\hat{x} = p\lambda \hat{x} \geq p\omega_h$. So, p constitutes a Walrasian equilibrium. QED □