The core

Lecture 7

November 15, 2015

Core

- We study a simple exchange economy that can be depicted in an Edgeworth-Bowley box.

- There are two goods two agents A and B with endowments $\omega_A = (\omega_{A1}, \omega_{A2})$ and $\omega_B = (\omega_{B1}, \omega_{B2})$.

- An allocation $x = (x_A, x_B)$ is feasible if $x_A + x_B = \omega_A + \omega_B$.

- Contract curve is the set of all Pareto optimal allocations.

- The core is the set of Pareto optimal allocations that are individually rational.

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- Assume that there are more than 2 agents, say, N agents.

- We say that an allocation $x = (x_1, x_2, ..., x_N)$ is blocked by coalition $S \subseteq N$ if there is another allocation y such that $y_i \succeq x_i$ for all $i \in S$ with at least one strict preference and $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.

- The core is the set of feasible allocations that are not blocked.

- Assume that in the economy there are two agents identical to A and two agents identical to B.

Core

- It is clear that if allocation x is in the core agents $A1$ and $A2$ cannot get different bundles that are equally good if their preferences are strictly convex.

- More interesting is that they have to get exactly the same bundles.

- Assume that they get different bundles and that $A1$ gets a strictly worse bundle than $A2$.

- Assume that $B1$ does not get a strictly better bundle than $B2$.

- Let us study coalition $\{A1, B1\}$.

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- The average of $B1$'s bundle and $B2$'s bundle is certainly at least as good as $B1$'s bundle

$$\frac{1}{2}x_{B1} + \frac{1}{2}x_{B2} \succeq_B x_{B1}$$

- The average of $A1$'s bundle and $A2$'s bundle is strictly better than $A1$'s bundle

$$\frac{1}{2}x_{A1} + \frac{1}{2}x_{A2} \succ_A x_{A1}$$

- But $\frac{1}{2}x_{B1} + \frac{1}{2}x_{B2} + \frac{1}{2}x_{A1} + \frac{1}{2}x_{A2} = \frac{1}{2}\left(x_{B1} + x_{B2} + x_{A1} + x_{A2}\right) = \frac{1}{2}\left(2\omega_B + 2\omega_A\right) = \omega_B + \omega_A$ is feasible to the coalition.

Core

- In a two-agent economy the worst core allocation from $A1$'s point of view is $h = (h_A, h_B)$ where h_A is on the same indifference curve as his/her endowment.
- In the four-agent economy it is not possible that A-type agents get h_A in the core.
- Assume to the contrary.
- Consider coalition $\{A1, A2, B1\}$.
- In the Edgeworth-Bowley box draw a line that connects h and $\omega = (\omega_A, \omega_B)$.
- Any allocation on the line is preferred to ω_A by A-types.
- Consider $k = \frac{1}{2}h + \frac{1}{2}\omega$.

Core

- Consider allocation that gives k_A to $A1$ and $A2$ and h_B to $B1$.
- Clearly A-types fare strictly better than at h and $B1$ does equally well.
- The resources the coalition uses are given by

$$2k_A + h_B = 2\left(\frac{1}{2}h_A + \frac{1}{2}\omega_A\right) + h_B = h_A + \omega_A + h_B$$

- This is feasible since $h = (h_A, h_B)$ is in the core of the two-agent economy and consequently

$$h_A + h_B = \omega_A + \omega_B$$

- It is clear that allocations 'close' to h do not belong to the core of the four-agent economy.

Core

- Any line from ω to an efficient allocation that is not the Walrasian equilibrium allocation f must cut either type of agent's indifference curve that passes through f.

- If f is to the south-west from the equilibrium allocation it cuts type-A agent's indifference curve so that it is above it close to f.

- So there is a point close to f preferred to f by type-A agents.

- Denote it by $k_A = \frac{1}{n}\omega_A + \frac{n-1}{n} f_A$ for some n.

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- A coalition where there are n type-A agents and $n-1$ type-B agents improves upon f.
- This happens by giving k_A to the A-type agents and f_B to B-type agents.
- The resources used are

$$nk_A + (n-1)f_B = \omega_A + (n-1)f_A + (n-1)f_B$$

$$= \omega_A + (n-1)(f_A + f_B)$$

$$= \omega_A + (n-1)(\omega_A + \omega_B)$$

- This means that f is not in the core.

Core

Example

Figuring out the core.

Let there be two consumers A and B.

The relevant data are: $\omega_A = (2,3)$, $\omega_B = (4,5)$, $u_A(x,y) = x^2 y$ and $u_B(x,y) = xe^y$.

To determine the core in the Edgeworth-Bowley box notice first that when A consumes x and y, B consumes $6-x$ and $8-y$.

The slope of A's indifference curve is given by $-\frac{u_y}{u_x} = -\frac{x}{2y}$.

The slope of B's indifference curve is given by $-\frac{u_y}{u_x} = x-6$.

Condition $-\frac{x}{2y} = x-6$ which is equivalent to $y = -\frac{x}{2x-12}$ gives the contract curve.

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Example

Consumer A can guarantee utility 12.
Bundle (x, y) on the contract curve gives the same utility if $x^2\left(-\frac{x}{2x-12}\right) = 12$ or $x \approx 3,76906$.
Consumer B can guarantee utility $4e^5$.
Bundle $(6-x, 8-y)$ on the contract curve gives the same utility if $x \approx 4,72641$.
Consequently the core is given by $f : [3,76906, 4,72641] \longmapsto \mathbb{R}$, $f(x) = -\frac{x}{2x-12}$.

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- The reasoning before the example hints that when the economy grows the core might go towards the Walrasian equilibrium allocation.

- The following example shows that something more is needed.

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Example

In the economy there are two consumers and two goods.
Each consumer has preferences $u(x,y) = (x+1)(y+1)$.
Endowments are given by $\omega_1 = (3,0)$ and $\omega_2 = (0,3)$.
Consider an increasing sequence of economies $\mathscr{E}_n$ such that in the nth economy there is one consumer of type 1 with endowment $\omega_{1n} = (3n,0)$ and n consumers of type 2 $\omega_{2n} = (0,3)$.
The Walrasian equilibrium allocation in the nth economy is given by $x_{1n} = \left(\frac{3n}{2}, \frac{3n}{2}\right)$ and $x_{2n} = \left(\frac{3}{2}, \frac{3}{2}\right)$.
But the core consists of allocations that give consumers of type 1 allocation $(n\alpha, n\alpha)$ and consumer of type 2 $(3-\alpha, 3\alpha)$ where $\frac{2}{3} \leq \alpha \leq 2$.

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Example

Let us see how this comes about by showing that $f_1 = (2n, 2n)$ for the consumer of type 1 and $f_2 = (1,1)$ for consumers of type 2 is in the core for all n.

Suppose this is not the case.

Then there exists a coalition S that can improve upon it, and consumer 1 has to belong to it.

Denote the improving allocation by w_a, $a \in S$.

Note that if there were prices and they were $p = (1,1)$ then the allocation would maximise the consumers' utility in the sets

$$\{z : pz \leq p(2n, 2n)\}$$

$$\{z : pz \leq p(1,1)\}$$

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Example

Now the following have to hold simultaneously

$$w_a \succ_a f_a$$

$$\sum_{a \in S} w_a = \sum_{a \in S} \omega_a$$

$$p \sum_{a \in S} w_a > p \sum_{a \in S} f_a$$

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Example

Let there be $k+1$ consumers in S which means that

$$p\sum_{a\in S} f_a = 2k+4n \geq p\sum_{a\in S}\omega_a = 3k+3n$$

But then

$$p\sum_{a\in S} w_a > p\sum_{a\in S}\omega_a$$

which is a contradiction.