General equilibrium theory

Lecture 5

November 8, 2015

GET basics

- The subject has a long history starting perhaps from Leon Walras. For a concise history I suggest that you have a look at section 1.3 here http://www.econ.ucsd.edu/~rstarr/113Winter2012/2010Chap1.pdf
- The aim of GET is to determine the conditions under which there exists an economywide equilibrium; all markets clear simultaneously.
- One is also interested in the properties of the equilibrium.
- The equilibrium concept is called competitive equilibrium or Walras-equilibrium.
- The presentation of the basic results draws heavily on Maskin and Roberts (2008, Economic Theory).

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- There is a set of consumers $\mathscr{H} = \{1, ..., H\}$ with preferences $\succeq_h$ or $u_h : \mathbb{R}^L_+ \to \mathbb{R}$.
- Each consumer has an endowment $\omega_h \in X_h$ where $X_h \subseteq \mathbb{R}^L_+$ is the consumption set.
- The set of commodities is given by $\mathscr{L} = \{1, ..., L\}$.
- This describes a pure exchange economy $\mathscr{E} = ((u_h, \omega_h)_{h \in \mathscr{H}})$.
- If we further postulate a set of firms $\mathscr{F} = \{1, ..., F\}$ with production sets $Y_f \subseteq \mathbb{R}^L$ we have an economy with production.

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- A generalised competitive mechanism associates with each price vector $p \in \triangle^{L-1}$ and each production plan $y_f \in Y_f$ income $I_h(p, \{y_f\})$ to each consumer $h \in \mathscr{H}$.
- In a Walrasian economy $I_h(p, \{y_f\}) = p\omega_h + \sum_{f \in \mathscr{F}} \theta_{fh} p y_f$ where θ_{fh} is the ownership share of consumer h of firm f.

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Definition

A Walras-equilibrium is a price $p \in \mathbb{R}^L_+$ such that

$$x_h(p) = argmax_w \left\{ u_h(w)\, s.t.\, pw \leq p\omega_h + \sum_{f \in \mathscr{F}} \theta_{fh} p y_f \right\}$$

and

$$y_f(p) = argmax_v \{ pv\, s.t.\, v \in Y_f \}$$

and

$$\sum_{h \in \mathscr{H}} x_h(p) = \sum_{f \in \mathscr{F}} y_f(p) + \sum_{h \in \mathscr{H}} \omega_h$$

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- A central question is the existence of Walras-equilibrium.
- This depends on the parameters of the problem.
- We assume that the preferences $\succeq_h$ can be represented by a utility function u_h that satisfies A1-A3 below.
- We also require that the endowment vectors are strictly positive.

1. A1 u_h is continuous, $h \in \mathscr{H}$.
2. A2 u_h is strictly increasing or $u_h(x) > u_h(y)$ whenever $x > y$, $h \in \mathscr{H}$.
3. A3 u_h is concave, $h \in \mathscr{H}$.
4. A4 $\omega_h \gg 0$, $h \in \mathscr{H}$.
5. A5 Y_f is closed and convex, and $0 \in Y_f$, $f \in \mathscr{F}$.

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- A2 guarantees that $px_h = I_h$ even if p is not an equilibrium price.
- Summing over all consumers one comes up with the Walras's law
$$p\left(\sum_{h\in\mathscr{H}}(x_h-\omega_h)-\sum_{f\in\mathscr{F}}y_f\right)=0$$
- A3 means that generally the excess demand is a correspondence.
- Let us denote it by $Z(p)=\{z\,|\,z=\sum_{h\in\mathscr{H}}(x_h-\omega_h)-\sum_{f\in\mathscr{F}}y_f\}$ where x_h maximises u_h subject to the budget constraint $h\in\mathscr{H}$, and y_f maximises profit subject to the technology and prices $f\in\mathscr{F}$.

Lemma. Z is well-defined, upper hemi-continuous, convex-valued, compact-valued and satisfies the Walras's law.

Proof.

This is somewhat complicated (see Debreu 1959). □

Lemma. If $p_l = 0$, then for all $z \in Z(p)$ it is the case that $z_l > 0$.

Proof.

Follows from the strict monotonicity of preferences. □

- The proof of existence requires a fixed point theorem.

Theorem

Kakutani's fixed point theorem. Let $X \subseteq \mathbb{R}^k$ be a convex and compact set. Let $f : X \longrightarrow X$ be a correspondence which is non-empty, convex-valued and upper hemi-continuous for all $x \in X$. There exists an $x \in X$ such that $x \in f(x)$.

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Theorem

Walras-equilibrium exists when A1-A5 are satisfied.

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Proof.

Consider p such that $p_l = 0$. Now $z_l > 0$ whenever $z \in Z(p)$. Upper hemi-continuity of Z means that there exists $\delta > 0$ such that for all p, for all $l \in \mathscr{L}$, it is the case that $z_l > 0$ whenever $p_l < \delta$ and $z \in Z(p)$. This further implies that there exists $K > 0$ such that for all p, for all $z \in Z(p)$, and all $l \in \mathscr{L}$, it is the case that $z_l + Kp_l > 0$. This is because the excess demand is bounded from below.

Let us define

$$H(p) = \left\{ w \,\middle|\, w = \frac{z + Kp}{\sum_l (z_l + Kp_l)}\ z \in Z(p) \right\}$$

which is a correspondence from $\triangle^{L-1}$ to itself. Clearly H satisfies the requirements of the Kakutani fixed point theorem because Z satisfies them, and there exists $\bar{p}$ such that $\bar{p} \in H(\bar{p})$. □

Proof.

Consider excess demand that corresponds to the fixed point, $\bar{z} \in Z(\bar{p})$ or

$$\bar{p} = \frac{\bar{z} + K\bar{p}}{\sum_l (\bar{z}_l + K\bar{p}_l)} \quad (1)$$

Now, if $\bar{p}_k = 0$ then $\bar{z}_k > 0$, and this is a contradiction with (1), and consequently $\bar{p}_l > 0$ for all $l \in \mathscr{L}$. If $\sum_{k \in \mathscr{L}} \bar{z}_k > 0$ then $\bar{z}_l > 0$ for all $l \in \mathscr{L}$; otherwise $K\bar{p}_l$ would be divided by something greater than K in (1) and the fixed point property would not hold (note that $\sum_l \bar{p}_l = 1$). But in this case the Walras's law does not hold. Analogously, we can discard the possibility that $\sum_{k \in \mathscr{L}} \bar{z}_k < 0$. But this means that $\sum_{k \in \mathscr{L}} \bar{z}_k = 0$ and $\bar{z}_l = 0$ for all $l \in \mathscr{L}$ at price $\bar{p}$. QED □

Definition

A feasible allocation $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$ is Pareto-efficient if for some feasible allocation $\left(\{\widetilde{x}_h\}_{h\in\mathscr{H}}, \{\widetilde{y}_f\}_{f\in\mathscr{F}}\right)$ there exists $i \in \mathscr{H}$ such that $\widetilde{x}_i \succ x_i$ then there exists $j \in \mathscr{H}$ such that $\widetilde{x}_j \prec x_j$.

Theorem

First welfare theorem. If preferences are strictly monotone then Walras-equilibrium is Pareto-efficient.

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Proof.

Assume that p is a Walras-equilibrium and $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$ a corresponding allocation. Contrary to the claim assume that there exists a feasible allocation $\left(\{\widetilde{x}_h\}_{h\in\mathscr{H}}, \{\widetilde{y}_f\}_{f\in\mathscr{F}}\right)$ that Pareto dominates $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$. We must have
$\sum_{h\in\mathscr{H}} x_h(p) = \sum_{f\in\mathscr{F}} y_f(p) + \sum_{h\in\mathscr{H}} \omega_h$ and
$\sum_{h\in\mathscr{H}} \widetilde{x}_h(p) = \sum_{f\in\mathscr{F}} \widetilde{y}_f(p) + \sum_{h\in\mathscr{H}} \omega_h$ and $p\widetilde{x}_h \geq px_h$ for all $h \in \mathscr{H}$ with at least one strict inequality; note that in the second relation we can assume equality because of strict monotonicity of preferences. □

Proof.

Sum over all consumers to get

$$\sum_{h\in\mathscr{H}} p\widetilde{x}_h(p) > \sum_{h\in\mathscr{H}} px_h(p)$$

Profit maximisation implies that $p\widetilde{y}_f(p) \leq py_f(p)$ for all $f \in \mathscr{F}$, and summing over all firms yields $\sum_{f\in\mathscr{F}} p\widetilde{y}_f(p) \leq \sum_{f\in\mathscr{F}} py_f(p)$. Combining in an evident way

$$p\left(\sum_{h\in\mathscr{H}} x_h(p) - \sum_{f\in\mathscr{F}} y_f(p) - \sum_{h\in\mathscr{H}} \omega_h\right) <$$

$$p\left(\sum_{h\in\mathscr{H}} \widetilde{x}_h(p) - \sum_{f\in\mathscr{F}} \widetilde{y}_f(p) - \sum_{h\in\mathscr{H}} \omega_h\right)$$

But this is a contradiction. QED □

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- Usually in the proof of the second welfare theorem also the existence of equilibrium is shown; typically using the separating hyperplane theorem.
- However, if the existence of equilibrium is assumed the proof is as simple as that of the first welfare theorem.

Theorem

Second welfare theorem. Assume that preferences are strictly monotone. Let $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$ *be Pareto efficient. Assume that consumer h gets income* $I_h = px_h$*. If an equilibrium exists, then* $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$ *is an equilibrium allocation.*

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Proof.

Assume that $\widetilde{p}$ is a Walras-equilibrium and $\left(\{\widetilde{x}_h\}_{h\in\mathscr{H}}, \{\widetilde{y}_f\}_{f\in\mathscr{F}}\right)$ the corresponding equilibrium allocation which is known to be Pareto efficient. As the income of any consumer h allows him/her to afford both $\widetilde{x}_h$ and x_h s/he must be indifferent between them, and since preferences are strictly monotone $px_h = p\widetilde{x}_h$. Profit maximisation implies that $\widetilde{p}y_f \leq \widetilde{p}\widetilde{y}_f$ for all $f \in \mathscr{F}$, and if for some firm the inequality is strict summing we get

$$\widetilde{p}\left(\sum_{h\in\mathscr{H}} \widetilde{x}_h - \sum_{f\in\mathscr{F}} \widetilde{y}_f\right) < \widetilde{p}\left(\sum_{h\in\mathscr{H}} x_h - \sum_{f\in\mathscr{F}} y_f\right)$$

□

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Proof.

But Pareto efficiency of the allocations and strict monotonicity of preferences means that

$$\sum_{h\in\mathscr{H}} x_h - \sum_{f\in\mathscr{F}} y_f = \sum_{h\in\mathscr{H}} \widetilde{x}_h - \sum_{f\in\mathscr{F}} \widetilde{y}_f = \sum_{h\in\mathscr{H}} \omega_h$$

and consequently all firms make the same profit as at allocation $\left(\{x_h\}_{h\in\mathscr{H}}, \{y_f\}_{f\in\mathscr{F}}\right)$; this must be an equilibrium allocation at prices $\widetilde{p}$. QED □