Decision making under uncertainty

Lecture 4

November 3, 2015

Example

Insurance

DM has wealth w and faces loss L with probability p.

DM can buy insurance that costs r per unit; a unit pays out unity in case of loss.

DM's problem is

$$max_x pu(w - L - rx + x) + (1 - p)u(w - rx) \tag{1}$$

The first order condition is given by

$$pu'(w - L - rx + x)(1 - r) - (1 - p)u'(w - rx)r = 0 \tag{2}$$

Examples

Example

For a risk averse DM this is a necessary and sufficient condition for the optimal solution $x^*(p, L, r)$.
One immediately sees that for an actuarially fair insurance, or $r = p$, the solution is given by $x^* = L$.
For some other price there is over or under insurance.

Examples

Example

From 2 one finds by total differentiation

$$\frac{dx}{dr} =$$

$$\frac{pu''(w-L+(1-r)x)x(1-r)+pu'(w-L+(1-x)r)-(1-p)u''(w-rx}{pu''(w-L+(1-x)r)(1-r)^2+(1-p)u''(w-rx)} \quad (3)$$

Thus, if the insurer is a monopoly its maximisation problem is

$$max_r rx - px$$

Examples

Example

The first order condition is

$$x+(r-p)\frac{dx}{dr}=0 \tag{4}$$

This gives some indication why usually the actuarially fair price is assumed.
The LHS of 4 evaluated at $r=p$ is given by $x>0$.
Thus, the monopoly would like to raise the price.

Example

Let us return to the assumption of a fixed price, and see how DM's wealth affects things.
Differentiating (2) with respect to wealth one gets expression

$$pu''(w-L+(1-r)x)(1-r)-(1-p)u''(w-rx)r$$

Evaluate this at x^* dividing the first term by $pu'(w-L-rx+x)(1-r)$ and the second term by $(1-p)u'(w-rx)r$ to get an expression of the same sign

$$\frac{u''(w-L+(1-r)x)}{u'(w-L+(1-r)x)}-\frac{u''(w-rx)}{u'(w-rx)}$$

The sign is the difference between Arrow-Pratt measure of risk aversion at wealth levels $w-rx$ and $w-L+(1-r)x$.
If the DM has decreasing absolute risk aversion the amount of insurance goes down when wealth increases.

Examples

- Risk aversion is a local measure but people tend to use it like a global one.
- Assuming that a person is risk averse everywhere leads so some weird consequences.

Example

Assume that a DM rejects an even gamble to win 11 and to lose 10. The s/he should reject any even gamble to win X and to lose 1000 for any X.

Example

To see the logic assume that initial wealth is w.
Rejecting the gamble is equivalent to

$$u(w+11)-u(w)<u(w)-u(w-10)$$

or the worth of the eleventh unit is at most $10/11$ of the worth of the tenth unit lost.
Or $u'(w+11) \leq \frac{u(w+11)-u(w)}{11} \leq \frac{10}{11}\frac{u(w)-u(w-10)}{10} \leq \frac{10}{11}u'(w-10)$.
Changes in the wealth in interval $[-10,11]$ are associated with a loss of about 10% in marginal utility.
If a DM rejects the above gamble at all wealth levels then going up in 21 unit steps marginal utility keeps decreasing by more than $10/11$.
As the marginal utility diminishes faster than a geometric series this implies that there is not finite amount X that the DM would accept in an even gamble with a loss of 1000.

Examples

Example

This as a critique of expected utility theory has a couple of weak points.

The first is evident, namely that the result relies on rejecting a certain gamble at all wealth levels; this might not be observable in practice.

The second is the idea that the DM's preferences are over final wealth levels.

von Neumann-Morgenstern axiomatisation, however, is silent about the outcomes of the gambles; they might as well be changes in relative wealth.

Rabin's absurd results are easily avoided by allowing more flexible utility functions.

Example

Perhaps more interesting is the Ellsberg paradox.
There are two urns with red and blue balls.
In urn-1 there 50 red balls and 50 blue balls.
In urn-2 there are altogether 100 red and/or blue balls but nothing more is know about their numbers.
Gamble g_r^1 gives 50 if a random draw from urn-1 is a red ball and zero otherwise.
Gamble g_b^1 gives 50 if a random draw from urn-1 is a blue ball and zero otherwise.
Gamble g_r^2 gives 50 if a random draw from urn-2 is a red ball and zero otherwise.
Gamble g_b^2 gives 50 if a random draw from urn-2 is a blue ball and zero otherwise.

Examples

Example

In hypothetical situations as well as in experiments people are indifferent between g_r^1 and g_b^1, as well as g_r^2 and g_b^2.
But they prefer g_r^1 to g_r^2, as well as g_b^1 to g_b^2.
This seems to lead to the conclusion that in gamble g_r^2 the proportion of red balls is less than $\frac{1}{2}$, and the proportion of blue balls is less than $\frac{1}{2}$.
The DM's attitude towards gambles is known as ambiguity averseness.

Examples

Example

Kreps 6.8. Assume that consumer's wealth is y, there are two goods, and von Neumann-Morgenstern preferences are given by $u(x,y) = f(x+y)$ where f is an increasing.
Consider two prices $p = (1,3)$ and $p' = (3,1)$.
We show that the consumer prefers risky situation where the prices p and p' with equal probabilities to price $\frac{1}{2}p + \frac{1}{2}p' = (2,2)$.
When each good costs 2 the consumer does not care which one to buy and gets utility $f\left(\frac{y}{2}\right)$.
When prices are either p or p' consumer buys the cheaper good and gets utility $f(y)$.

Examples

Example

Assume next that the preferences are given by $f(min\{x,y\})$ where f is concave and strictly increasing, $f(0)$ finite.

The possible prices can be $p = (\gamma, \gamma)$ or $p' = \left(\frac{1}{\gamma}, \frac{1}{\gamma}\right)$ where $\gamma > 1$.

We show that for fixed γ one can find function f such that consumer prefers prices $\frac{1}{2}p + \frac{1}{2}p'$ to the risky situation.

Also for a fixed f one can find γ large enough such that the consumer prefers prices $\frac{1}{2}p + \frac{1}{2}p'$ to the risky situation.

As prices are equal consumer spends half his/her income on each good and gets utility $\frac{1}{2}f\left(\frac{y}{2\gamma}\right) + \frac{1}{2}f\left(\frac{y\gamma}{2}\right)$ in the risky situation and

$f\left(\frac{y}{2\left(\frac{1}{2}\gamma + \frac{1}{2}\frac{1}{\gamma}\right)}\right)$ in the non-risky situation.

Let γ grow without bound; In the latter expression argument goes to zero in the former this happens only to the first term.

Examples

Example

Fix γ and notice that $\frac{y}{2\gamma} < \frac{y}{\gamma+\frac{1}{\gamma}} < \frac{y\gamma}{2}$.

Consider a piecewise linear function $f(x) = x$ for $x \leq \frac{y}{\gamma+\frac{1}{\gamma}}$,

$f(x) = \frac{y}{\gamma+\frac{1}{\gamma}} + \varepsilon\left(x - \frac{y}{\gamma+\frac{1}{\gamma}}\right)$ for $x > \frac{y}{\gamma+\frac{1}{\gamma}}$.