Decision making under uncertainty

Third lecture

November 1, 2015

State dependent utility

- In games one needs a utility representation for settings where the utility depends on different states of the world.
- It is easy to imagine, for instance, that DM's utility depends on whether s/he is ill or healthy.
- To achieve this representation one enlarges the space where preferences are defined.
- Let X be the set of outcomes, and $\triangle(X)$ the set of simple lotteries.
- Let there be a finite set of states Ω.
- Key role is played by the set of random variables Λ; $g \in \Lambda$ is a mapping $g : \Omega \to \triangle(X)$.

State dependent utility

- In state $\omega \in \Omega$ random variable g is mapped to a simple lottery $l_g(\cdot|\omega) \in \Delta(X)$.
- Evaluated at outcome x one gets the conditional probability given ω, $g(\omega)(x) = l_g(x|\omega)$ or the probability of outcome x at state ω.
- Clearly $a\lambda_1 + (1-a)\lambda_2 \in \Lambda$ for $\lambda_1, \lambda_2 \in \Lambda$ and $a \in (0,1)$.
- Finally, it is assumed that there exists a probability measure π over Ω; $\pi(\omega)$ is DM's belief about the likelihood of ω.
- Imposing the von Neumann-Morgenstern axioms on the preferences $\succeq$ over Λ we get the following result.

State dependent utility

Theorem

Let Ω be finite and let $\succeq$ on Λ be complete, transitive and continuous, and satisfy independence. Then there exists $u : X \times \Omega \to \mathbb{R}$ such that if $g, h \in \Lambda$ then $h \succeq g$ if and only if

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{x \in X} l_h(x|\omega)\, u(x,\omega) \geq$$

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{x \in X} l_g(x|\omega)\, u(x,\omega)$$

Function $v = a + bu$ represents the same preferences as long as $a \geq 0$ and $b \in \mathbb{R}$, and this is the most general transformation allowed .

State dependent utility

- With state dependent utility it is possible to generate more variation to DM's choices than before.
- In insurance context if different accidents correspond to different states then the DM does not necessarily fully insure even though the price of insurance is actuarially fair.

State dependent utility

- The above differs from axiomatisations where one gets representations with subjective probabilities.

- Action a may yield utility in either of the following state dependent ways

$$U(a) = \sum_{\omega \in \Omega} u_\omega (a(\omega))$$

$$U(a) = \sum_{\omega \in \Omega} p(\omega) u(a(\omega))$$

  where the latter representation is usually used.

- Assume two equally likely states $\Omega = \{\omega_1, \omega_2\}$ where ω_1 signifies that it is sunny and ω_2 that it is rainy.

- DM can take either of two actions; $a_1 =$umbrella if raining and suntan lotion is sunny, $a_2 =$umbrella if sunny and suntan lotion if rainy.

State dependent utility

- The second representation makes either action equally good.
- It is usually thought that 'umbrella when rainy' $\succ$'lotion when rainy', and 'lotion when sunny' $\succ$'umbrella when sunny'.
- In the first representation with this ordering the first act is better than the second act.
- The second representation is achieved by assuming that the values of the prizes are independent of the state.
- The second representation is used much more because one gets the probabilities out.
- In most applications the prizes consist of money, and state dependency is not a big issue.

State dependent utility

Example

Individual's utility depends on wealth w and health $h \in \{0,1\}$ by $u(w,h)$.
We assume $u_1 > 0$, $u_2 > 0$ and $u_{11} < 0$.
Initially the level of wealth is w and $h = 1$.
With probability p the individual falls ill; then s/he incurs loss L and changes health status to $h = 0$.
Insurance is offered at a fair price per unit $\pi = p$.
The individual's problem is given by

$$max_x pu(w - px - L + x, 0) + (1 - p)u(w - px, 1)$$

State dependent utility

Example

The first order condition to the problem is given by

$$u'(w - px - L + x, 0) - u'(w - px, 1) = 0$$

If $u(.,0) = u(.,1)$ it is immediate that the individual takes full insurance $x = L$.
If $u'(y,0) < u'(y,1)$ the at full insurance the LHS of the FOC is negative.
Then decreasing x makes $u'(w - px, 1)$ smaller and $u'(w - px - L + x, 0)$ bigger.
Consequently, the individual chooses less than full insurance.

Complete ordering of gambles

- Stochastic dominances and most other orders over the set of lotteries are very partial.
- Let us shortly consider an approach where one gets a complete ordering.
- This is based on Foster and Hart (JPE, 2009).
- There is a closely related approach by Aumann and Serrano (JPE, 2008).

Complete ordering of gambles

- One has to restrict the underlying set of gambles a little, as well as the preferences of the DMs.
- The set of gambles that are considered is such that they have positive expectation and both negative and positive outcomes.
- A DM prefers non-bankrupty to bankrupty.
- Too risky gambles are rejected; this is related to the DM's wealth.
- Riskiness of a gamble is the level of wealth such that DM is indifferent between accepting and rejecting the gamble.

Complete ordering of gambles

- Each gamble has finitely many possible realisations.
- The set of possible gambles is finitely generated.
- DM's wealth changes in time which is discrete.
- Initial wealth is W_1.
- Each period t DM is offered a gamble which s/he can accept or reject.
- If s/he accepts his/her wealth in period $t+1$ is $W_{t+1} = W_t + g_t$.
- The gambles that are offered do not come from any distribution; non-Bayesian approach.

Complete ordering of gambles

- A critical wealth function $Q(g)$ gives a minimal wealth level for a gamble g to be acceptable.
- $Q(g)$ depends only on g.
- $Q(\lambda g) = \lambda Q(g)$.
- A simple strategy for a DM with wealth W is if $W \geq Q(g)$ accept g, if $W < Q(g)$ reject g.
- We say that a sequence of lotteries induces bankruptcy if $W_t = 0$ for some t or $lim_{t \to \infty} W_t = 0$.
- It induces no bankruptcy if $Pr(lim_{t \to \infty} W_t = 0) = 0$.
- A strategy guarantees no bankruptcy if for all wealth levels and all sequences of gambles $Pr(lim_{t \to \infty} W_t = 0) = 0$.

Complete ordering of gambles

Theorem

For every lottery there exists a unique real number $R(g) > 0$ such that a simple strategy with a critical wealth function Q guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$. Moreover, $R(g)$ is determined by

$$E\left(log\left(1+\frac{1}{R(g)}g\right)\right)=0$$

Corollary

A simple strategy that guarantees no-bankruptcy is given by "reject g if and only if $W < R(g)$".

Complete ordering of gambles

- $R(g)$ is the measure of riskiness and it satisfies homogeneity of degree one, subadditivity and convexity.
- If lottery g first order stochastically dominates h then $R(g) < R(h)$.
- If lottery g second order stochastically dominates h then $R(g) < R(h)$.
- There is also an axiomatisation for R.
- Surprisingly, the defining condition does not have solution for lotteries with infinite support (Riedel & Hellmann, 2013).

Complete ordering of gambles

- Consider gamble with equally likely outcomes $a > 0 > b$ and $a > -b$.
- It is immediate that $E\left(log\left(1 + \frac{1}{R(g)}g\right)\right) = \frac{1}{2}log\left(1 + \frac{1}{R(g)}a\right) + \frac{1}{2}log\left(1 + \frac{1}{R(g)}b\right) = 0$ implies $R(g) = -\frac{ab}{a+b}$.
- Inserting $a = 120$ and $b = -100$ gives $R(g) = 600$; inserting $a = 105$ and $b = -100$ gives $R(g) = 2100$.
- $R(g)$ can be interpreted as a minimal reserve to participate in gamble g.

Complete ordering of gambles

- Let us see how the continuous distribution case is problematic.

- Assume that a gamble is uniformly distributed on $[-100, 200]$.

- Then the defining equation is given by

$$\phi(\lambda) = \int_{-100}^{200} \frac{1}{300} log(1 + \lambda x)\, dx$$

- The derivative is given by $\phi'(\lambda) = \int_{-100}^{200} \frac{1}{300} \frac{x}{1+\lambda x} dx$.

- The second derivative is given by
$\phi''(\lambda) = -\int_{-100}^{200} \frac{1}{300} \frac{x^2}{(1+\lambda x)^2} dx < 0$

Complete ordering of gambles

- At $\lambda = 0$ $\phi'(0) = \int_{-100}^{200} \frac{1}{300} x dx > 0$
- Evaluating the integral at $\lambda = \frac{1}{100}$ one needs to go to limit.

$$\phi(\lambda) = \int_{-100}^{200} \frac{1}{300} log(1 + \lambda x) dx =$$

$$\frac{1}{300} log(1 + \lambda x) x \rrbracket_{-100}^{200} - \frac{1}{300} \int_{-100}^{200} \frac{\lambda x}{1 + \lambda x} dx$$

To do the last integral make the integrand $1 - \frac{1}{1+\lambda x}$.

- When $\lambda \longrightarrow \frac{1}{100}$ problematic terms vanish.
- One should get that $\phi\left(\frac{1}{100}\right) = log 3 - 1 > 0$.