Decision making under uncertainty

Lecture 2

October 27, 2015

Risk

- The concept of risk comes up in a multitude of instances.
- In economics it is, in the end, the curvature of the Bernoulli utility function.
- DMs are most of the time assumed to dislike risk as in regarding a certain income of 100 as better than a lottery that pays 200 or zero with equal probabilities.
- We proceed by considering only the set of lotteries M whose outcomes are in money.
- One of the basic points is that expectation of a lottery is not a good utility measure.

Risk

- The expectation of lottery $m \in M$ is given by $E(m) = \int_{-\infty}^{\infty} y m(y) dy$.
- Its expected utility is given by $U(m) = \int_{-\infty}^{\infty} u(y) m(y) dy$.
- We want that for a risk averse DM $U(m) = \int_{-\infty}^{\infty} u(y) m(y) dy < U(E(m)) = u(E(m))$.
- For the simple lottery above $u(100) > \frac{1}{2} u(200) + \frac{1}{2} u(0)$.
- But this is pretty much the definition of a concave function u.

Risk

Definition

A DM is i) risk averse if u is concave or $u'' < 0$ in differentiable world, ii) risk loving if u is convex or $u'' > 0$ in a differentiable world, iii) risk neutral if u is affine or $u'' = 0$ in a differentiable world.

- Notice that we can apply the definition at a point $x \in \mathbb{R}$ or require that it holds for all $x \in \mathbb{R}$.
- In the former case attitude towards risk is local; in the latter case it is global.

Risk

- To see what aspects are important for the everyday use of the concept of risk consider a lottery Y with expectation $E(Y)$.
- The Taylor-approximation of expected utility $E(u(Y))$ evaluated at $E(Y)$ is given by

$$E(u(Y)) \approx$$

$$E\left(u(E(Y)) + u'(E(Y))(Y - E(Y)) + \frac{1}{2}u''(E(Y))(Y - E(Y))^2\right) =$$

$$u(E(Y)) + u'(E(Y))(E(Y) - E(Y)) + \frac{1}{2}u''(E(Y))E\left((Y - E(Y))^2\right) =$$

$$u(E(Y)) + \frac{1}{2}u''(E(Y))\,Var(Y)$$

Risk

- Thus, the difference between the expected utility and the utility of the expectation is

$$u(E(Y)) - E(u(Y)) \approx -\frac{1}{2}u''(E(Y))\,Var(Y)$$

- It depends on the curvature of the utility function and the variance of the lottery.
- A DM is defined to be risk averse if s/he always prefers a degenerate lottery $\int_{-\infty}^{\infty} xdF(x)$ to the lottery F.
- Arrow-Pratt coefficient of absolute risk aversion is given by $r_A(x,u) = -\frac{u''(x)}{u'(x)}$.
- The larger it is the more risk averse a DM is at a particular point/wealth level x.
- Notice that the curvature is divided by the derivative to keep it invariant to affine transformations.

Risk

- There are equivalent ways of describing risk aversion.
- Denote the certainty equivalent of lottery F by $u(c(F,u)) = \int_{-\infty}^{\infty} u(x)dF(x)$.

Theorem

A DM is risk averse $\Leftrightarrow$ *u is concave and increasing* $\Leftrightarrow$ $c(F,u) \leq \int_{-\infty}^{\infty} x dF(x)$ *for all lotteries* F.

Proof.

Assume a DM is risk averse. Thus, for any lottery F $\int_{-\infty}^{\infty} u(x)dF(x) \leq u\left(\int_{-\infty}^{\infty} x dF(x)\right)$. This is called Jensen's inequality and it characterises concave functions. Suppose then that u is concave and increasing. Again Jensen's inequality and the definition of $c(F,u)$ give us $u(c(F,u)) = \int_{-\infty}^{\infty} u(x)dF(x) \leq u\left(\int_{-\infty}^{\infty} x dF(x)\right)$ from which it is immediate that $c(F,u) \leq \int_{-\infty}^{\infty} x dF(x)$. Finally, assume that $c(F,u) \leq \int_{-\infty}^{\infty} x dF(x)$. This is equivalent to $u(c(F,u)) \leq u\left(\int_{-\infty}^{\infty} x dF(x)\right)$ which is, by the definition of certainty equivalence, equivalent to $\int_{-\infty}^{\infty} u(x)dF(x) \leq u\left(\int_{-\infty}^{\infty} x dF(x)\right)$. □

The coefficient of absolute risk aversion measures what it aims by the following proposition.

Theorem

$r_A(x, u_2) \geq r_A(x, u_1)$ if and only if there exists a concave function ψ such that $u_2 = \psi(u_1)$.

Proof.

There exists an increasing function ψ such that $u_2 = \psi(u_1)$. Differentiating $u_2'(x) = \psi'(u_1(x))\, u_1'(x)$ and further $u_2''(x) = \psi''(u_1(x))\,(u_1'(x))^2 + \psi'(u_1(x))\, u_1''(x)$. Dividing sideways we get $\frac{u_2''(x)}{u_2'(x)} = \frac{\psi''(u_1(x))(u_1'(x))^2 + \psi'(u_1(x))u_1''(x)}{\psi'(u_1(x))u_1'(x)}$ which is equivalent to $r_A(x, u_2) = r_A(x, u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))} u_1'(x)$. Now $r_A(x, u_2) \geq r_A(x, u_1)$ if and only if $\psi'' \leq 0$. □

Risk

- Other equivalent statements can be found in the text book.

Definition

The coefficient of relative risk aversion is defined as $r_R(x,u) = x r_A(x,u)$.

- The idea is that risk aversion depends on income, and it is usually postulated that $r_R(x,u)$ decreases with wealth or $-\frac{u''}{u'} - x\frac{u'''u' - (u'')^2}{(u')^2} < 0$.
- Risk aversion coefficients tell how a DM looks at risk locally but usually they are used globally by requiring that one DM's coefficient is greater than another one's at all wealth levels.
- When this does not happen one cannot order the DMs.

Risk

- One might also want to provide an ordering for lotteries.

Definition

A lottery F first order stochastically dominates lottery G if $\int_{-\infty}^{\infty} u(x)dF \geq \int_{-\infty}^{\infty} u(x)dG$ for any non-decreasing function $u : \mathbb{R} \to \mathbb{R}$.

Fact

First order stochastic dominance is equivalent to the following condition: $F(x) \leq G(x)$ for all x.

Definition

Assume that lotteries F and G have the same mean. We say that lottery F second order stochastically dominates lottery G if $\int_{-\infty}^{\infty} u(x)dF \geq \int_{-\infty}^{\infty} u(x)dG$ for any non-decreasing function $u : \mathbb{R} \to \mathbb{R}$.

- Once again there are many equivalent conditions for second order stochastic dominance.
- Let X be a lottery with distribution F, and let Z be a symmetric zero mean lottery.
- Lottery $Y \equiv X + Z$ has the same mean as X but larger variance.
- Denote the distribution of Y by G.
- Now F second order stochastically dominates G.
- We say that is got from F by a mean preserving spread.

Theorem

Let lotteries F and G be continuous and have the same mean with support on interval $[\underline{x}, \bar{x}]$. F second order stochastically dominates G if and only if $\int_{\underline{x}}^{x} F(t)dt \leq \int_{\underline{x}}^{x} G(t)dt$ for all x.

Proof.

Let us assume that u is twice differentiable, and that there exist $\underline{x}$ such that $F(\underline{x}) = G(\underline{x}) = 0$, and $\bar{x}$ such that $F(\bar{x}) = G(\bar{x}) = 1$. Partially integrating

$$\int_{\underline{x}}^{\bar{x}} u(x)dF - \int_{\underline{x}}^{\bar{x}} u(x)dG = |_{\underline{x}}^{\bar{x}} u(x)(F(x) - G(x)) + \int_{\underline{x}}^{\bar{x}} u'(x)(G(x) - F(x)$$

$$= \int_{\underline{x}}^{\bar{x}} u'(x)(G(x) - F(x))\,dx \quad (1)$$

□

Proof.

As F and G have the same mean $\int_{\underline{x}}^{\bar{x}} xdF = \int_{\underline{x}}^{\bar{x}} xdG$, and partially integrating this $|_{\underline{x}}^{\bar{x}} xF(x) - \int_{\underline{x}}^{\bar{x}} F(x)dx = |_{\underline{x}}^{\bar{x}} xG(x) - \int_{\underline{x}}^{\bar{x}} G(x)dx$ or $\int_{\underline{x}}^{\bar{x}} F(x)dx = \int_{\underline{x}}^{\bar{x}} G(x)dx$. Next integrate partially (1) to get

$$|_{\underline{x}}^{\bar{x}} u'(x) \left(\int_{\underline{x}}^{x} G(t)dt - \int_{\underline{x}}^{x} F(t)dt \right) - \int_{\underline{x}}^{\bar{x}} u''(x) \left(\int_{\underline{x}}^{x} G(t)dt - \int_{\underline{x}}^{x} F(t)dt \right)$$

$$= - \int_{\underline{x}}^{\bar{x}} u''(x) \left(\int_{\underline{x}}^{x} G(t)dt - \int_{\underline{x}}^{x} F(t)dt \right) dx \qquad (2)$$

As $u'' < 0$ we see that if $\int_{\underline{x}}^{x} F(t)dt \leq \int_{\underline{x}}^{x} G(t)dt$ then (2) is negative and F is preferred to G. □

Risk

Proof.

Assume then that $\int_{\underline{x}}^{\widetilde{x}} F(t)dt > \int_{\underline{x}}^{\widetilde{x}} G(t)dt$ for some $\widetilde{x}$, and let $u(x) = x$ for $x \leq \widetilde{x}$,and $u(x) = \widetilde{x}$ for $x > \widetilde{x}$. Then u is concave and

$$\int_{\underline{x}}^{\bar{x}} u(x)dF(x) - \int_{\underline{x}}^{\bar{x}} u(x)dG(x) =$$

$$\int_{\underline{x}}^{\widetilde{x}} xdF(x) - \int_{\underline{x}}^{\widetilde{x}} xdG(x) + \tilde{x}\left(G\left(\tilde{x}\right) - F\left(\tilde{x}\right)\right) =$$

$$\Big|_{\underline{x}}^{\widetilde{x}} x\left(F(x) - G(x)\right) - \int_{\underline{x}}^{\widetilde{x}} \left(G\left(\tilde{x}\right) - F\left(\tilde{x}\right)\right) + \tilde{x}\left(G\left(\tilde{x}\right) - F\left(\tilde{x}\right)\right) =$$

$$\int_{\underline{x}}^{\widetilde{x}} \left(G\left(\tilde{x}\right) - F\left(\tilde{x}\right)\right) < 0$$

□