Decision making under uncertainty

Lecture 1

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- Most economically interesting problems cannot be understood, or they do not exist, in a world without uncertainty.
- Von Neumann and Morgenstern developed the, nowadays, standard approach while working on the theory of games.
- It usually goes by the name of expected utility.
- There are other approaches but vNM-theory is the easiest.

- There is a set of outcomes $X = \{x_1, x_2, ..., x_n\}$.
- There is a set of gambles or lotteries $\Delta(X)$ over the outcomes.
- $p = (p_1, p_2, ..., p_n) \in \Delta(X)$ is a probability distribution where p_i is the probability that outcome x_i is realised.
- We naturally require that $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$.
- The exposition is easier assuming that X is finite but one can derive the results also for infinite sets of outcomes.
- Then one has to assume that there is the worst and the best outcome in X.

- Outcome x_i can be identified with a degenerate lottery where $p_i = 1$.
- A convex combination of two lotteries can be regarded as a lottery: $\lambda p + (1-\lambda)q \in \Delta(X)$ if $p, q \in \Delta(X)$ and $\lambda \in [0,1]$, where the probability of outcome x_i is $\lambda p_i + (1-\lambda)q_i$.
- Convex combinations of degenerate lotteries are called simple lotteries.
- Convex combinations of lotteries as well as convex combination of convex combinations of lotteries (and so on) are called compound lotteries.
- It is assumed that a DM can always reduce a compound lottery to a corresponding simple lottery and regards these as equally good.

Axioms

- Decision maker has a complete and transitive preference relation $\succeq$ over $\Delta(X)$

  A1. $\succeq$ is continuous, or for $p, q, r \in \Delta(X)$ such that $p \succeq r \succeq q$ there exists $\lambda \in [0,1]$ such that $\lambda p + (1-\lambda)q \sim r$.

  A2. $\succeq$ satisfies independence, or for $p, q, r \in \Delta(X)$ and $\lambda \in [0,1]$ $p \succeq q$ if and only if $\lambda p + (1-\lambda)r \succeq \lambda q + (1-\lambda)r$.

Definition

Utility function $U : \Delta(X) \to \mathbb{R}$ is of expected utility form if there are numbers $(u_1, u_2, ..., u_n)$, where u_i is the utility associated with outcome x_i, such that for $p \in \Delta(X)$ the utility is given by $U(p) = \sum_{i=1}^{n} p_i u_i$. Note: U is necessarily continuous.

- One can think that the utility numbers are generated by utility function $u : X \to \mathbb{R}$.
- It is called a Bernoulli utility function.
- A utility function that has the expected utility form is linear in probabilities or $U(\lambda p + (1-\lambda)r) = \lambda U(p) + (1-\lambda)U(r)$.

Theorem

$\succeq$ is a binary relation on $\Delta(X)$ satisfying completeness, transitivity, continuity and independence if and only if there exists a utility function $U : \Delta(X) \to \mathbb{R}$ that represents $\succeq$ and U is of expected utility form.

Proof.

Assume that U is of expected utility form. It is clear that $\succeq$ is complete, transitive and continuous. To show independence assume that $p \succeq r$. Then $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} r_i u_i$ which is equivalent to $\lambda \sum_{i=1}^{n} p_i u_i \geq \lambda \sum_{i=1}^{n} r_i u_i$ for $\lambda \in [0,1]$. This is equivalent to $\lambda \sum_{i=1}^{n} p_i u_i + (1-\lambda) \sum_{i=1}^{n} q_i u_i \geq \lambda \sum_{i=1}^{n} r_i u_i + (1-\lambda) \sum_{i=1}^{n} q_i u_i$. But this is equivalent to $\lambda p + (1-\lambda) q \succeq \lambda r + (1-\lambda) q$. □

Proof.

Assume that $\succeq$ satisfies the stated properties. The proof is constructive, and to that end consider the best element $\bar{p}$ and the worst element $\underline{p}$ in $\Delta(X)$. If $\bar{p} \sim \underline{p}$ there is nothing to prove as one can choose U to be any constant. So assume that $\bar{p} \succ \underline{p}$. Consider any $p \in \Delta(X)$. By continuity there exists $\lambda(p) \in [0,1]$ such that $p \sim \lambda(p)\bar{p} + (1 - \lambda(p))\underline{p}$. (SHOW THAT THIS IS UNIQUE). Now we have a candidate for the utility function, namely $U(p) = \lambda(p)$. Assume that $p \succeq q$. This is equivalent with $\lambda(p)\bar{p} + (1 - \lambda(p))\underline{p} \succeq \lambda(q)\bar{p} + (1 - \lambda(q))\underline{p}$ which is equivalent with $\lambda(p) \geq \lambda(q)$ (FOLLOWS FROM UNIQUENESS ABOVE). HOW? □

Proof.

To show that U has the expected utility form consider a convex combination $\mu p + (1-\mu)q$ of lotteries p and q. It is known that $p \sim U(p)\bar{p} + (1 - U(p))\underline{p}$ and $q \sim U(q)\bar{p} + (1 - U(q))\underline{p}$. Consequently, $\mu p + (1-\mu)q \sim$ $\mu \left[U(p)\bar{p} + (1 - U(p))\underline{p}\right] + (1-\mu)\left[U(q)\bar{p} + (1 - U(q))\underline{p}\right]$ where we have cheated a little and applied independence twice simultaneously. HOW? Rearranging we find that the last expression is in relation $\sim$ to $(\mu U(p) + (1-\mu)U(q))\,\bar{p} + (1 - \mu U(p) - (1-\mu)U(q))\,\underline{p}$. By the definition of U we must have $U(\mu p + (1-\mu)q) = \mu U(p) + (1-\mu)U(q)$. □

Theorem

Assume that U represents preference $\succeq$ on $\Delta(X)$ and has the expected utility property. Then also $V : \Delta(X) \to \mathbb{R}$ represents $\succeq$ and has the expected utility property if and only if $V(x) = a + bx$ where $a, b \in \mathbb{R}$ and $b > 0$.

Proof.

Assume first that V represents $\succeq$ and has an expected utility form. Consider $p \in \Delta(X)$, and remember that there exists $\lambda(p)$ such that $p \sim \lambda(p)\bar{p} + (1 - \lambda(p))\,\underline{p}$. Because both U and V have the expected utility representation
$U(p) = \lambda(p)U(\bar{p}) + (1 - \lambda(p))\,U\left(\underline{p}\right)$ and
$V(p) = \lambda(p)V(\bar{p}) + (1 - \lambda(p))\,V\left(\underline{p}\right)$. From the first expression we can solve $\lambda(p) = \frac{U(p) - U\left(\underline{p}\right)}{U(\bar{p}) - U(\underline{p})}$. Inserting this to the latter expression and reorganising one sees that $V(p) = a + bU(p)$ where
$a = V\left(\underline{p}\right) - U\left(\underline{p}\right)\frac{V(\bar{p}) - V\left(\underline{p}\right)}{U(\bar{p}) - U\left(\underline{p}\right)}$ and $b = \frac{V(\bar{p}) - V\left(\underline{p}\right)}{U(\bar{p}) - U\left(\underline{p}\right)}$. □

Proof.

Assume then that $V = a + bU$. It is immediate that $U(q) \geq U(p)$ implies that $V(q) \geq V(p)$ or that V represents $\succeq$. Next define $v_i = a + bu_i$ and notice that
$V(p) = a + bU(p) = a + b\sum_{i=1}^{n} p_i u_i = \sum_{i=1}^{n} p_i (a + bu_i) = \sum_{i=1}^{n} p_i v_i$
so that V has the expected utility property. □

- In the von Neumann-Morgenstern representation theorem probabilities are regarded as objective.
- One can also start from the set up which is more primitive and derive the probabilities from the decision maker's preferences.
- These are called subjective probabilities.
- This is how Savage's approach works.
- In the end the representation theorems are pretty much the same in the form.
- There is not much difference either as far as single-person decision making is concerned.
- In strategic situations it is typically assumed that the players share a common view about probabilities.
- This is hard to motivate if the players are assumed to have utility functions that come up in Savage's framework.

Savage's approach

- Rudiments of Savage's approach follow.
- There is a set of states (of the world) S and a set of outcomes or consequences X.
- Decision maker can choose acts that are functions $f : S \to X$; an act could be a purchase of an insurance policy.
- If there are sufficiently many acts available the DM's preferences over the acts allow enough order to elicit probabilities.
- The idea is to impose some (rationality) requirements on the way the DM orders acts.
- Given a slightly longer list of requirements/axioms than in the vNM-theory one can show that the DM has a utility representation that is formally similar to that of vNM.

- In particular, there exists a function $p : S \to \mathbb{R}_+$ that can be regarded as a probability distribution over S, and a function $u : X \to \mathbb{R}$ that can be regarded as a utility function.
- For the DM the following are equivalent for acts f and g: $f \succeq g$ and $\sum_{s \in S} u(f(s))\, p(s) \geq \sum_{s \in S} u(g(s))\, p(s)$.
- Notice that for the representation both the probability function p and the utility function u are derived from preferences; they are not independent.
- For instance, one cannot change the utility function and see what happens if the DM becomes more risk averse.
- That is because changing u also changes p.
- This has not prevented economists from doing exactly this.