FDPE Microeconomics 1: Decision theory
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Problem Set 3

1. Let production function $f : \mathbb{R}^K_+ \to \mathbb{R}_+$ reflect homothetic technology: $f(\cdot) = g(h(\cdot))$ for functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R}^K_+ \to \mathbb{R}_+$ such that g is monotonically increasing and h is homogenous of degree 1 : $h(ty) = th(y)$ for all $t > 0$. Show that $f(y) = f(y')$ implies $f(ty) = f(ty')$.

2. Let production function be of CES form $f(y_1, y_2) = (y_1^\rho + y_2^\rho)^{1/\rho}$, for $\rho > 0$. Compute the associated cost function.

   (a) Argue that Leontief $\min\{y_1, y_2\}$, linear y_1+y_2, and Cobb-Douglas $y_1 y_2$ production functions can be interpreted as special cases of CES.

   (b) Show that $f(y_1, y_2) = (y_1^\rho + y_2^\rho)^{1/\rho}$.satisfies *constant elasticity of substitution* $\sigma(y)$ at any y (guess where the acronym CES comes from)
$$\sigma(y) = \frac{d\ln(y_2/y_1)}{d\ln(\frac{f_1(y)}{f_2(y)})} = \frac{1}{1-\rho}$$
   where $\partial f(y)/\partial y_k = f_k(y)$.

3. Production function has increasing (constant, decreasing) returns to scale at $y \in \mathbb{R}^K_+$, if $f(ty) > tf(y)$ $(=, <)$ for all $t > 1$.

   (a) Let the production function be Cobb-Douglas $f(y_1, y_2) = \beta y_1^{\alpha_1} y_2^{\alpha_2}$. Under which parameter $\alpha_1, \alpha_2, \beta > 0$ values does it have increasing, constant, or decreasing returns?

   (b) Let $f(y_1, y_2)$ have constant returns to scale. Show that if the average product of y_1 is increasing, then the marginal product of y_2 is decreasing.

4. *Output elasticity* at y reflects the presentage increase in output when all inputs are increased by one presentage:
$$\varepsilon(y) = \frac{df(ty)}{dt}\frac{t}{f(ty)}\bigg|_{t=1}.$$

Output elasticity of factor k at y reflects the presentage increase in output when input k is increased by one presentage:

$$\varepsilon_i(y) = \frac{df(y)}{dy_i}\frac{y_i}{f(y)}.$$

(a) Show that $\varepsilon(y) = \sum_{k=1}^{K} \varepsilon_i(y)$.

(b) Compute $\varepsilon_i(y)$ for the Cobb-Douglas production function $f(y) = \beta y_1^{\alpha_1} y_2^{\alpha_2}$.

5. Let $y(p,w)$ be the input demand function. Show that $\partial y_k(p,w)/\partial w_k \leq 0$ and that $\partial y_k(p,w)/\partial w_j = \partial y_j(p,w)/\partial w_k$, for all inputs j,k.

6. Let $\pi(p,w)$ be the profit function of the firm. Show that:

   (a) If $p \geq p'$ and $w_k \leq w'_k$ for all $k = 1, ..., K$, then $\pi(p,w) \geq \pi(p',w')$.

   (b) π is homogenous of degree 1: $\pi(tp,tw) = t\pi(p,w)$, for all t.

7. Compute the profit function associated to the production function $f(y) = y^\alpha$, where $y \in \mathbb{R}_+$ and $0 < \alpha < 1$. Verify that the profit function is homogenous and convex in (p,w). Find the optimal production level.

8. Compute the input demand functions and the supply function associated to the production function $f(y_1,y_2) = \beta \ln y_1 + (1-\beta)\ln y_2$, where $y_1, y_2 \in \mathbb{R}_+$ and $0 < \beta < 1$.

9. A real valued function g is *superadditive* if $g(y_1 + y_2) \geq g(y_1) + g(y_2)$, for all y_1, y_2. Show that every cost function is superadditive in input prices. Use this to prove that the cost function is nondecreasing in input prices without requiring it to be differentiable.

10. Let $c(w,q)$ be the cost function of the firm. Show step-by-step (using the envelope argument) that the associated Lagrange multiplier equals the marginal cost of production $\partial c(w,q)/\partial q$.