Lecture notes 4: Theory of production

Hannu Vartiainen
HECER

Fall 2015

Producer theory

- Start with a single firm facing given prices
- Need to describe the technology of the firm
- **Exogenous**: prices
- **Endogenous**: output and input demands
- Aim to understand the optimal production decision of the firm
  - No attention to organizational nor stratgic aspects
  - Objective to have a model that can be transferred in it its pure form to the general equilibrium framework
- Key difference to the consumer model
  - no income effects
  - everything observable

Primitives:

- Firm with one production good in $\mathbb{R}_+$
- Input space $\mathbb{R}^K$
- The primitive of the model: **production function**

$$f : \mathbb{R}^K_+ \to \mathbb{R}_+$$

describes the output/input combinations that are **technologically feasibe**

Axiom

Production function f is continuous, increasing, and quasiconcave

- By monotonicity, if $y \geq y'$, then $f(y) \geq f(y')$
- By quasiconcavity, the input requirement set $V(x) = \{y \in \mathbb{R}^K_+ : f(y) \geq x\}$ is *convex* for all $x \in \mathbb{R}_+$
- Firm's production function can be represented by the **production possibility set**

$$Y = \{(y, x) \in \mathbb{R}^{K+1}_+ : f(y) \geq x\}$$

- Continuous, increasing, and quasiconcave production function corresponds to a production possibility set Y that is
  - convex: if $(y, x), (y', x') \in Y$, then $\lambda(y, x) + (1 + \lambda)(y', x') \in Y$ for all λ
  - monotonic: $y \in V(x)$ and $y' \geq y$ imply $y \in V(x)$
  - closed

Profit maximization problem

- With output price $p > 0$ and input prices $w = (w_1, ..., w_K)$, where $w_k > 0$ for all k, **profit** is

$$pf(y) - w \cdot y$$

  for any $y \in \mathbb{R}_+^K$ (use the dot product notation $w \cdot y = \sum_k w_k y_k$)
- The firm's objective is to **maximize profits,** i.e. to maximize the size of the owner's budget set

Optimal production

- The problem reduces to

$$\max_{y \in \mathbb{R}_{+}^{K-1}} pf(y) - w \cdot y \tag{1}$$

- Letting $y(p, w)$ denote the optimal choice(s) at prices (p, w),

$$\pi(p, w) = pf(y(p, w)) - w \cdot y(p, w)$$

  is the **profit function** of the firm
- Assuming differentiable f and an interior solution ($y_k > 0$ for all $k = 1, .., K$), the FOC associated to the optimum $y(p, w)$ is

$$\frac{\partial f(y(p, w))}{\partial y_k} - \frac{w_k}{p} = 0, \qquad \text{for all inputs } k$$

- **Marginal rate of technological substitution**:

$$MRTS_{kj} = \frac{\partial f\left(y(p,w)\right)/\partial y_k}{\partial f\left(y(p,w)\right)/\partial y_j}, \quad \text{for all inputs } k,j$$

  signifies the slope of the **isoquant** $\{y \in \mathbb{R}_+^{K-1} : f(y) = q\}$ at $q = f\left(y(p,w)\right)$

- At the optimum,

$$MRTS_{kj} = \frac{w_k}{w_j}, \quad \text{for all inputs } k,j$$

- Suppose one observes the profit of the firm and prices, can we deduce the production function and the optimal production?
- Using the envelope argument:

Proposition (Hotelling's Lemma)

$$\frac{\partial \pi(p,w)}{\partial p} = f(y(p,w))$$

$$\frac{\partial \pi(p,w)}{\partial w_k} = -y_k(p,w), \quad \textit{for all inputs } k$$

- Hence the profit function $\pi(p,w)$ is decreasing in w_k, increasing in p

- A function $g : \mathbb{R}^K \to \mathbb{R}_+$ is convex if $tg(x) + (1-t)g(x') \geq g(tx + (1-t)x')$ for all $x, x' \in \mathbb{R}^K$, for all $t \in (0,1)$

Proposition

The profit function $\pi(p,w)$ is a convex in (p,w)

Proof.

First, for any (p,w), (p',w') and (p'',w''),

$$\begin{aligned}\pi(p,w) &= pf\left(y(p,w)\right) - w\cdot y(p,w) \\ &\geq pf\left(y(p'',w'')\right) - w\cdot y(p'',w'')\end{aligned}$$

and

$$\begin{aligned}\pi(p',w') &= p'f\left(y(p',w')\right) - w'\cdot y(p',w') \\ &\geq p'f\left(y(p'',w'')\right) - w'\cdot y(p'',w'')\end{aligned}$$

□

Proof.

(cont.) Hence

$$
\begin{aligned}
& t\pi(p,w) + (1-t)\pi(p',w') \\
\geq\ & t[pf\left(y(p'',w'')\right) - w \cdot y(p'',w'')] \\
& +(1-t)[p'f\left(y(p'',w'')\right) - w' \cdot y(p'',w'')] \\
=\ & [tp + (1-t)p']f\left(y(p'',w'')\right) \\
& -[tw + (1-t)w'] \cdot y(p'',w'')
\end{aligned}
$$

Since this holds for any (p'',w'') it holds in particular when $(p'',w'') = t(p,w) + (1-t)(p',w')$, which gives the result. □

- Using Hotelling's Lemma, by the convexity of π,

$$\frac{\partial f(y(p,w))}{\partial p} = \frac{\partial^2 \pi(p,w)}{(\partial p)^2} \geq 0$$

$$\frac{\partial y_k(p,w)}{\partial w_k} = -\frac{\partial^2 \pi(p,w)}{(\partial w_k)^2} \leq 0, \quad \text{for all } k = 1, ..., K-1$$

- Interpretation:
  - If the price of an output increases, then the supply increases: "Law of Supply"
  - If the price of an input increases, the demand for the input decreases: "Law of Input Demand"

- Note that $f(y(\cdot, w))$ defines firm's optimal output under any output price p, i.e. $f(y(\cdot, w))$ is the **firm's supply function**
- Since

$$\pi(p, w) = \int_0^p \frac{\partial \pi(p', w)}{\partial p} dp' = \int_0^p f(y(p', w)) dp'$$

firm's profits $\pi(p, w)$ represented by the area between the output price axis and $f(y(\cdot, w))$, until p

Cost minimization

- For each quantity of output, q, find the least costly input combination that yields q:

$$\min_{y \in \mathbb{R}_+^{K-1}} w \cdot y$$

$$\text{s.t. } q = f(y)$$

- Denote the solution by $z(w, q)$, i.e. the **conditional factor demand** function
- The value function, $c(w, q)$, is called the **cost function**

$$c(w, q) = w \cdot z(w, q).$$

- $z(w, q)$ is completely analogous to $h(p, u)$ in consumer theory and $c(w, q)$ is analogous to $e(p, u)$

Proposition

The cost function $c(w,q)$ is increasing in q, concave in w, increasing in p, and homogenous of degree one.

Proof.

We show that c is increasing in q. Note that $c(w,q)$ minimizes the Lagrangian

$$\mathcal{L}(w,q) = w \cdot y - \lambda[f(y) - q].$$

Hence

$$c(w,q) = w \cdot z(w,q) - \lambda[f(z(w,q)) - q].$$

By the envelope theorem

$$\frac{\partial c(w,q)}{\partial q} = \lambda$$

□

Proof.

(cont.) Since, at the optimum,

$$w_k = \lambda \frac{\partial f(z(w,q))}{\partial y_k}$$

it follows that

$$\frac{\partial c(w,q)}{\partial q} = \left(\frac{\partial f(z(w,q))}{\partial y_k}\right)^{-1} w_k \geq 0$$

□

...back to optimal production

- Given the notion of cost function, the problem of the firm simplies remarkably: just find the optimal level of output!
- That is, given p and w, the firm's objective is to solve

$$\max_{q \in \mathbb{R}_+} pq - c(w, q)$$

- The first order condition for this is the familiar:

$$p = \frac{\partial c(w, q)}{\partial q}$$

  i.e., at the firm's optimum, the **marginal cost** equals the price
- Thus the marginal cost curve $\partial c(w, \cdot) / \partial q$ defines the firm's **inverse supply function**

- Since

$$pq - c(w,q) = \int_0^q \left(p - \frac{\partial c(w,q')}{\partial q} \right) dq'$$

we see that the area between p and the inverse supply curve reflects the firm's profit

- Since $f(y(\cdot, w))$ the the supply function, we conclude that $p = \partial c(w, f(y(p,w)))/\partial q$, for all p: **duality in production**

Geometry of costs

- Assume that the firm has to invest $K > 0$ to operate in the market
- Sunk costs do not affect the optimal production (assuming that participating the market is profitable)
- Denote the average (minimum) costs of the firm from production q by

$$AC(q) = \frac{K + c(w, q)}{q}$$

- Then $AC(\cdot)$ is decreasing whenever the marginal cost $\partial c(w, q) / \partial q$ is lower than $AC(q)$ and increasing when $\partial c(w, q) / \partial q$ is higher than $AC(q)$ => the curves cross at the minimum of $AC(\cdot)$
- Unless the firm can operate at price p such that $p = \partial c(w, f(y(p, w))) / \partial q \geq AC(f(y(p, w)))$, it does not enter the market

Difference between consumer and producer theory:

- The utility function u only represents preferences $\succsim$ and cannot be observed even in principle: multiple respresentations
- Production function f is a unique description of the technology and, in principle, observable
- Conclusion: Not only ordinal but also cardinal differences have meaning under f, e.g. concavity of f matters!