Lecture notes 3a: Optimal consumption, duality, and welfare

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Optimal consumption

- Our next task is to understand the properties of the optimal consumption
- Let consumer preferences $\succsim$ be *rational, monotonic, strictly convex, and continuous*
  - There is a continuous utility function u that represents $\succsim$
  - There is a **unique** optimizer in $B(p, w)$, referred as the **Marshallian demand** $x(p, w)$ under (p, w)
- To analyse the properties of $x(p, w)$, it useful to set up a formal optimizing programme through which $x(p, w)$ is generated

- Consumer's problem can be represented as a **constrained utility maximization problem**

$$\max_{x \in B(p,w)} u(x).$$

or, equivalently,

$$\max_{x \geq 0} u(x)$$
$$\text{s.t. } \sum_{\ell=1}^{L} p_\ell x_\ell \leq w.$$

- The solution $x(p, w)$ can now be obtained via standard techiniques
- Construct a Lagrangean

$$\mathcal{L}(x, \lambda) = u(x) - \lambda \left(\sum_{\ell=1}^{L} p_\ell x_\ell - w \right),$$

where $\lambda \in \mathbb{R}$ is the **Lagrange multiplier**

- Let $x(p, w) := x^* > 0$ maximize the Lagrangean (assuming that u is differentiable)
- The first order conditions (FOC) are

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_\ell} - \lambda p_\ell &= 0, \text{ for all } \ell \\ \sum_{\ell=1}^{L} p_\ell x_\ell^* - w &= 0 \end{aligned}$$

- Thus

$$\frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} = \frac{p_\ell}{p_k}$$

- The ratio p_ℓ/p_k is the **marginal rate of substitution** between goods ℓ and k at x^* : the rate at which x_ℓ should increase when x_k decreases (or vice versa) for the agent's utility to remain intact

Example

Let preferences be charcacterized by a Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}, \text{ where } \alpha \in (0,1)$$

We derive the Marshallian demand x_1^*, x_2^*. First order conditions for optimality:

$$\begin{aligned} \alpha \left(\frac{x_2^*}{x_1^*} \right)^{1-\alpha} - \lambda p_1 &= 0 \\ (1-\alpha) \left(\frac{x_2^*}{x_1^*} \right)^{-\alpha} - \lambda p_2 &= 0 \\ p_1 x_1^* + p_2 x_2^* &= w \end{aligned}$$

We have

$$\frac{x_2^*}{x_1^*} = \frac{(1-\alpha) p_1}{\alpha p_2}$$

Example

(cont.) and solving for x_1^* and x_2^*

$$x_1^* = \frac{\alpha w}{p_1}, \; x_2^* = \frac{(1-\alpha)w}{p_2}$$

Indirect utility

- We are mainly interested in understanding the effect of price changes on consumption and welfare
- Given prices p and income w, define the consumer's **indirect utility function** by

$$v\left(p,w\right)=u\left(x\left(p,w\right)\right),$$

where $x(p,w)$ is the Marshallian demand under (p,w)

- What are the properties of $v\left(p,w\right)$ implied by utility maximization?

- Let u represent monotonic and continuous preferences $\succsim$
- Then the indirect utility function $v(\cdot, \cdot)$ is:
  1. i homogenous of degree 0 ($v(p, w) = v(tp, tw)$ for all $t > 0$)
  2. ii strictly increasing in w, strictly decreasing in p_ℓ
  3. iii continuous
- An important tool in analysing the indirect utility (or any value function resulting from maximization) is the **envelope theorem:** only the **direct** effect of a parameter change matters when evaluating the effects of changes in the environment

- To see this, recall first that by the FOC of the associated Lagrangean,

$$\frac{\partial}{\partial x_\ell}\mathcal{L}(x(p,w),\lambda) = \frac{\partial u\left(x(p,w)\right)}{\partial x_\ell} - \lambda p_\ell = 0, \text{ for all } \ell$$

- Thus

$$\begin{aligned}
\frac{\partial v\left(p,w\right)}{\partial p_\ell} &= \frac{\partial}{\partial p_\ell}\mathcal{L}(x\left(p,w\right),\lambda) \\
&= \frac{\partial}{\partial p_\ell}\left[u\left(x\left(p,w\right)\right) - \lambda\left(\textstyle\sum_\ell p_\ell x_\ell\left(p,w\right) - w\right)\right] \\
&= \sum_k \frac{\partial x_k\left(p,w\right)}{\partial p_\ell}\left(\frac{\partial u\left(x(p,w)\right)}{\partial x_k} - \lambda p_k\right) - \lambda x_\ell\left(p,w\right) \\
&= -\lambda x_\ell\left(p,w\right)
\end{aligned}$$

where the third equality follows by the chain rule and the last one from the FOC of the Lagrangean

- Similarly,

$$
\begin{aligned}
\frac{\partial v(p,w)}{\partial w} &= \frac{\partial}{\partial w}\mathcal{L}\left(x(p,w),\lambda\right) \\
&= \frac{\partial}{\partial w}\left[u\left(x(p,w)\right) - \lambda\left(\textstyle\sum_\ell p_\ell x_\ell(p,w) - w\right)\right] \\
&= \sum_\ell \frac{\partial x_\ell(p,w)}{\partial w}\left(\frac{\partial u(x(p,w))}{\partial x_\ell} - \lambda p_k\right) + \lambda \\
&= \lambda
\end{aligned}
$$

  where the second equality follows by the chain rule and the last one from the FOC of the Lagrangean
- Thus Lagrange multiplier λ gives the marginal (shadow) value of relaxing the constraint, i.e. the marginal value of wealth w

- Since

$$\frac{\partial v(p,w)}{\partial p_\ell} = -\lambda x_\ell(p,w) \quad \text{and} \quad \frac{\partial v(p,w)}{\partial w} = \lambda$$

we have:

Proposition (Roy's Identity)

The Marshallian demand $x(p,w)$ can be recovered from the indirect utility function $v(p,w)$ by

$$x_\ell(p,w) = -\frac{\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w}$$

Example

(cont.) With Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, the Marshallian demand is

$$x_1(p, w) = \frac{\alpha w}{p_1}, \; x_2(p, w) = \frac{(1-\alpha)w}{p_2}$$

The indirect utlity is

$$\begin{aligned} v(p, w) &= x_1(p, w)^{\alpha} x_2(p, w)^{1-\alpha} \\ &= \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \\ &= w \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} \end{aligned}$$

Example

With **quasilinear utility function** $u(x, m) = v(x) + m$, v increasing, differentiable and concave, the optimization problem is

$$\max v(x) + m$$
$$\text{s.t. } \bar{p}x + m \leq \bar{w}$$

where $\bar{p}$ is the price ratio p_x / p_m and $\bar{w}$ is the ratio w / p_x.Marshallian demand depends only on $\bar{p}$ and $\bar{w}$ and hence we may denote it by $x(\bar{p}, \bar{w}), m(\bar{p}, \bar{w})$ (assume > 0). It satisfies

$$\begin{aligned} v'(x(\bar{p}, \bar{w})) &= \bar{p} \\ \bar{p}x(\bar{p}, \bar{w}) + m(\bar{p}, \bar{w}) &= \bar{w} \end{aligned}$$

Since v is a concave function, $x(\bar{p}, \bar{w})$ is a decreasing function of $\bar{p}$ (why?)

Duality

- The problem with the utility function and the indirect utility functions is that they are not observable, only x, p and w are
- An important property called **duality of consumption** transforms the problem into language of the observables, and hence allows us to make emprically testable predictions
- Given the utility function $u(\cdot)$, denote by $h\left(p, \bar{u}\right)$ the choice that solves the **expedinture minimizing** problem subject to the utility being at least $\bar{u}$ (a number):

$$\min_{x_\ell \geq 0} \sum_{\ell=1}^{L} p_\ell x_\ell$$
$$\text{s.t. } u\left(x\right) \geq \bar{u}$$

- Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}^L_{++}$

- Function $h(p, \bar{u})$ is called the **Hicksian** or **compensated demand:** it evaluaes the effects of prices on consumption *as if* the consumer is compensated the income needed to stay in the same indifference curve
- Denote the value function under the minimizer by $e(p, \bar{u})$, the **expenditure function**

$$e(p, \bar{u}) = \sum_{\ell=1}^{L} p_\ell h_\ell(p, \bar{u})$$

Proposition

The Hicksian demand function satisfies compensated law of demand: $(p_\ell - p'_\ell)(h_\ell(p', \bar{u}) - h_\ell(p', \bar{u})) \leq 0$, *for all* ℓ, *for any price vectors* p, p'.

- Since $e(p, \bar{u})$ minimizes costs under the constraint that utility $\bar{u}$ is generated, and since $\bar{u}$ can be generated under w such that $\bar{u} = v(p, w)$, we have

$$w \geq e(p, v(p, w))$$

- Similarly,

$$\bar{u} \leq v(p, e(p, \bar{u}))$$

- The duality between the indirect utility function $v(\cdot, \cdot)$ and the expenditure function $e(\cdot, \cdot)$ manifests itself in the following parity:

Proposition

Let continuous, monotonic, and strictly convex preferences be represented by the utility function u. For any price vector $p \in \mathbb{R}^L_{++}$,

$$w = e(p, v(p, w)) \text{ and } \bar{u} = v(p, e(p, \bar{u}))$$

Proof.

Suppose that $w > e(p, v(p, w))$. Then there is a less costly way to attain utility $u = v(p, w)$ than $x(p, w)$, say y. Thus $\sum_\ell p_\ell y_\ell < w$. But by strict convexity of preferences, $\lambda x(p, w) + (1-\lambda)y \succ x(p, w)$, for all $\lambda \in (0, 1)$. Moreover, since

$$\lambda \sum_\ell p_\ell x_\ell(p, w) \leq \lambda w \quad \text{and} \quad (1-\lambda)\sum_\ell p_\ell y_\ell < (1-\lambda)w$$

also

$$\sum_\ell p_\ell[\lambda x_\ell(p, w) + (1-\lambda)y_\ell] < w$$

and hence $\lambda x(p, w) + (1-\lambda)y$ belongs to the budget set. But this contradicts the assumption that $x(p, w)$ is an optimal choice. Similar argument rules out $\bar{u} < v(p, e(p, \bar{u}))$. □

- By strict convexity, the Marshallian demand $x(p, w)$ and the Hicksian demand $h(p, w)$ are uniquely defined at each (p, w)
- Thus we also obtain a parity between the Marshallian demand function $x(\cdot, \cdot)$ and the Hicksian demand function $h(\cdot, \cdot)$

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, \bar{u}) = x(p, e(p, \bar{u}))$$

- The underlying force behind the duality is that any two disjoint convex sets can be separated by a hyperplane => a minimizer of a linear function in one set is at least a maximizer of the other set

- A counterpart of Roy's Identity can now be stated in the context of expedinture functions and Hicksian demand functions

Proposition (Shephard's Lemma)

The Hicksian demand $h(p, w)$ can be recovered from the expenditure function $e(p, \bar{u})$ by

$$h_\ell(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_\ell}$$

- To see this, observe that $e(p, \bar{u})$ is the value of the Lagrangean

$$\mathcal{L}(x, \lambda) = \sum_{\ell=1}^{L} p_\ell x_\ell - \lambda[u(x) - \bar{u}]$$

at the minimizer $x = h(p, \bar{u})$

- By FOC,

$$\begin{aligned} p_\ell - \lambda \frac{\partial u(h(p, \bar{u}))}{\partial x_\ell} &= 0, \text{ for all } \ell \\ u(h(p, \bar{u})) - u &= 0 \end{aligned}$$

- Thus, using again the envelope argument,

$$
\begin{aligned}
\frac{\partial e(p,w)}{\partial p_\ell} &= \frac{\partial \mathcal{L}(h(p,\bar{u}),\lambda)}{\partial p_\ell} \\
&= \frac{\partial \left\{ \sum_{k=1}^{L} p_k h_k(p,\bar{u}) - \lambda [u(h(p,\bar{u})) - \bar{u}] \right\}}{\partial p_\ell} \\
&= h_\ell(p,\bar{u}) + \sum_{k=1}^{L} \frac{\partial h_k(p,\bar{u})}{\partial p_\ell} \left(p_k - \lambda \frac{\partial u(h(p,\bar{u}))}{\partial x_k} \right) \\
&= h_\ell(p,\bar{u})
\end{aligned}
$$

Example

(cont.2) With Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, the FOC for the Hicksian demand h^* is

$$\begin{aligned} p_1 - \lambda\alpha \left(\frac{x_2^*}{x_1^*}\right)^{1-\alpha} &= 0 \\ p_2 - \lambda(1-\alpha) \left(\frac{x_2^*}{x_1^*}\right)^{-\alpha} &= 0 \\ (x_1^*)^{\alpha}(x_2^*)^{1-\alpha} - \bar{u} &= 0 \end{aligned}$$

We have

$$\frac{h_2^*}{h_1^*} = \frac{(1-\alpha)p_1}{\alpha p_2}$$

and solving for x_1^* and x_2^*

$$h_1^* = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} \bar{u}, \;\; h_2^* = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} \bar{u}$$

Example

(cont.3) With Hicksian demand

$$h_1(p, \bar{u}) = \left(\frac{\alpha p_2}{(1-\alpha)p_1} \right)^{1-\alpha} \bar{u}, \;\; h_2(p, \bar{u}) = \left(\frac{(1-\alpha)p_1}{\alpha p_2} \right)^{\alpha} \bar{u}$$

The expedinture function

$$\begin{aligned} e(p, \bar{u}) &= p_1 \left(\frac{\alpha p_2}{(1-\alpha)p_1} \right)^{1-\alpha} \bar{u} + p_2 \left(\frac{(1-\alpha)p_1}{\alpha p_2} \right)^{\alpha} \bar{u} \\ & \left[\left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} \left(\frac{p_2^{1-\alpha}}{p_1^{-\alpha}} \right) + \left(\frac{(1-\alpha)}{\alpha} \right)^{\alpha} \left(\frac{p_1^{\alpha}}{p_2^{1-\alpha}} \right) \right] \bar{u} \\ &= \left(\frac{p_1}{\alpha} \right)^{\alpha} \left(\frac{p_2}{1-\alpha} \right)^{1-\alpha} \bar{u} \end{aligned}$$

- An interesting feature of the expedinture function is that it is **concave** in p

Proposition

For any prices p and p', and for any $\lambda \in (0,1)$,

$$\lambda e(p, \bar{u}) + (1-\lambda) e(p', \bar{u}) \leq e(\lambda p + (1-\lambda) p', \bar{u})$$

- In particular, concavity implies that $\partial^2 e / (\partial p_\ell)^2 \leq 0$ for all ℓ

Proof.

Since $h(p, \bar{u})$ minimizes costs to achieve $\bar{u}$ under p and $h(p', \bar{u})$ minimizes costs to achieve $\bar{u}$ under p' we have, for any p'',

$$\begin{aligned} \sum_\ell p_\ell h_\ell(p, \bar{u}) &\leq \sum_\ell p_\ell h_\ell(p'', \bar{u}) \\ \sum_\ell p'_\ell h_\ell(p', \bar{u}) &\leq \sum_\ell p'_\ell h_\ell(p'', \bar{u}) \end{aligned}$$

Since the inequalities hold for any p'', they hold particular for $p'' = \lambda p + (1 - \lambda)p'$. Multiplying the first inequality with λ and the second with $(1 - \lambda)$ and summing side by side, □

Proof.

(cont.)

$$\begin{aligned} & \lambda \sum_\ell p_\ell h_\ell(p, \bar{u}) + (1-\lambda) \sum_\ell p'_\ell x_\ell(p', \bar{u}) \\ \leq \quad & \sum_\ell (\lambda p_\ell + (1-\lambda) p'_\ell) h_\ell(\lambda p + (1-\lambda) p', \bar{u}) \end{aligned}$$

Since $\sum_\ell p_\ell h_\ell(p, \bar{u}) = e(p, \bar{u})$ etc., the result follows. □