Lecture notes 2: Utility and consumer choice

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Fall 2015

Consumer with preferences

- Up to now, all discussion concerning the economic agent has been completely general
- Now we turn to an economically important special case: the **consumer,** who makes choices over feasible combinations of commodities
- In these notes, we lie down the standard axioms imposed on consumer behavior and study their implications
- The model is used when we turn to modeling markets

- We take $X = \mathbb{R}^L_+$, the set of all possible combinations of L distinct commodities indexed by $\ell = 1, ..., L$
- An element $x_1, ..., x_L$ of X is called a **bundle,** where x_ℓ is the quantity of good ℓ
- In addition to those implied by rationality (transitivity, completeness), we impose some extra conditions on preferences that facilitate meaningful comparison between the bundles and guarantee the induced choice is "well behaved"

Monotonicity

- Monotonicity is the condition that gives the commodity the meaning of a "good": more is better

Axiom (Monotonicity)

Preferences $\succsim$ *are* **monotonic** *if, for all* $x, y \in X$,

$$\begin{aligned} x_\ell &> y_\ell, \text{ for all } \ell \text{ imply } x \succ y \\ x_\ell &\geq y_\ell \text{ for all } \ell \text{ imply } x \succsim y \end{aligned}$$

- It is important that monotonicity does not restrict preferences at all in cases where the quantity of at least one good decreases

- Graphically, monotonicity precludes the possibility that indifference set $I(x)$ such that

$$I(x) = \{y \in X : y \sim x\}$$

does not contain segment that "bends upward" and that $I(x)$ lies above $I(y)$ whenever $x \succ y$

Examples

Let $L = 2$. Monotonic preferences:

- $(x_1, x_2) \succsim (y_1, y_2)$ if $ax_1 + (1-a)x_2 \geq ay_1 + (1-a)y_2$, $a \in (0, 1)$ (linear)
- $(x_1, x_2) \succsim (y_1, y_2)$ if $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$ (Leontief)
- $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1^\alpha x_2^{1-\alpha} \geq y_1^\alpha y_2^{1-\alpha}$, $\alpha \in (0, 1)$ (Cobb-Douglas)
- $(x_1, x_2) \succsim (y_1, y_2)$ if $v(x_1) + x_2 \geq v(y_1) + y_2$, for increasing $v(\cdot)$ (quasi-linear)

- A function u on $\mathbb{R}^L_+$ is **nondecreasing** if for any two bundles x and y such that $x_i \geq y_i$ for all $i = 1, ..., L$, $u(x) \geq u(y)$

Proposition

Preferences $\succsim$ are monotonic if (and only if) the utility function u representing $\succsim$ is nondecresing.

- A weaker axiom with similar spirit:

Axiom (Nonsatiation)

Preferences $\succsim$ are **locally nonsatiated** *if for all $x \in X$ and for all $\delta > 0$, there exists $y \in X$ such that*

$$\|y - x\| < \delta \text{ and } y \succ x.$$

- Local nonsatiation is implied by monotonicity but not vice versa
- Graphically, local nonsatiation implies that the indifference set $I(x) = \{y \in X : y \sim x\}$ is in fact a curve, containing no $L-$dimensional balls

Convexity

- Rationality, continuity, and monotonicity guarantee that indifference curves are downward sloping but may have kinks, i.e. moving towards a preferred bundle may actually make the agent worse off
- The next condition guarantees this will never happen

Axiom (Convexity)

Preferences $\succsim$ *are* **convex** *if for all* $x, y, \in X$ *and for all* $\lambda \in [0, 1]$,

$$x \succsim y \text{ implies } (\lambda x + (1 - \lambda)y) \succsim y$$

They are **strictly convex** *if for all* $x, y, \in X$ *and for all* $\lambda \in (0, 1)$,

$$x \succsim y \text{ implies } (\lambda x + (1 - \lambda)y) \succ y$$

Convexity and the utility function

- Function $f : \mathbb{R}^L_+ \to \mathbb{R}$ is **concave** if $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ and **quasi-concave** if $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$, for all $x, y \in \mathbb{R}^L_+$ and $\lambda \in [0,1]$
- Concave function is quasi-concave but not necessariy vice versa
- Concavity of the utility function is an often made assumption in economics theory, what does it entail from the underlying preferences?

Proposition

Preferences $\succsim$ on $X = \mathbb{R}^L_+$ are convex if and only if a utility function u representing them is quasi-concave

Corollary

If the utility function u is concave, then the preferences u represents are convex

- If, in addition, the preferences a monotonic $\succsim$, then the utility function representing them is increasing and and quasi-concave, i.e. for x, y such that $u(x) = u(y)$ it holds that $\lambda u(x) + (1 - \lambda) u(y) \leq u(x)$

Indifference curves

- For continuous, convex and monotonic preferences $\succsim$ on $\mathbb{R}^L_+$, the indifference set $I(x) = \{y \in X : y \sim x\}$ satisfies the following properties:

1. $I(x)$ has measure zero, is a curve (nonsatiation)
2. strictly decreasing (monotonicity)
3. continuous (cont. + monotonicity)
4. convex (convexity)

$=>$ **a map of upwards opening, preference ranked indifference curves**

Characterization

- Recall the notion of continuous preferences

Axiom (Continuity)

Preferences $\succsim$ *are* **continuous** *if, for all* $x \in X$, *the upper and lower contour sets* $\succsim(x)$ *and* $\precsim(x)$ *are closed*

- Continuity implies that perturbing a budle slightly does not affect its standing in strict preferences
- Debreu's Theorem states the existence of a continuous utility function when preferences are continuous

- A useful way to construct a continuous utility function when preferences are also monotonic: find (exercise) a function $t : X \to \mathbb{R}_+$ such that $x \sim (t(x), ..., t(x))$ for all $x \in X$

Proposition

Let rational preferences on X be continuous, convex and monotonic. Then $u(x) = t(x)$ for all x represents the preferences. Moreover u is continuous.

Proof.

Let $x \succsim y$. Then $(t(x), ..., t(x)) \succsim (t(y), ..., t(y))$. By monotonicity $t(x) \geq t(y)$, i.e. $u(x) \geq u(y)$. For continuity, let $x^k \to x$ for some sequence $\{x^k\}$ of bundles. We show that also $u(x^k) \to u(x)$. Suppose that $t(x^k) \to t > t(x)$. Then there is high enough K such that $t(x^k)$ is in the $(t + t(x))/2$ -neighborhood of t. By the definition of $t(\cdot)$ and monotonicity of preferences,

$x^k \sim (t(x^k), ..., t(x^k)) \succ ((t + t(x))/2, ..., (t + t(x))/2)$, for all $k > K$. Since $x^k \to x$, by the continuity of the preferences, $x \succsim ((t + t(x))/2, ..., (t + t(x))/2)$. But since $(t + t(x))/2 > t(x)$, by monotonicity, also $x \succ (t(x), ..., t(x))$, contradicting the basic tenet that $(t(x), ..., t(x)) \sim x$. The direction $t < t(x)$ proceeds similarly. □

Quasi-linear preferences

- Applications in economics aften employ quasi-linear utility function, where the utility of the DM is linear in a designated commodity ("money")
- Formally, the outcome space is $X = Y \times \mathbb{R}_+$ where $x \in Y$ and $m \in \mathbb{R}_+$ induce a utility

$$u(x, m) = v(x) + m$$

  where $v : Y \to \mathbb{R}_+$ is a sub-utility function
- Preferences on $Y \times \mathbb{R}_+$ that this utility function characterizes have the following properties

1. For any $x \in Y$ and $m, m' \in \mathbb{R}_+$, $(x, m) \succsim (x, m')$ iff $m \geq m'$
2. For any $x, x' \in Y$ and $m, m', m'' \in \mathbb{R}_+$, $(x, m) \succsim (x', m')$ iff $(x, m + m'') \succsim (x', m' + m'')$

- By (2), there are **no weath effects,** i.e. the "initial" level of *m* does not affect the comparisons outcomes (other than *m*)
- Gives meaning to differential changes in payoffs
- But note: adding distinct agents' utilities together still not meaningful! (why?)

Consumer's problem

- We have constructed the consumer preferences on the set of consumption bundles $X = \mathbb{R}^L_+$
- We are mainly interested choices in "economic domains", where the consumer's feasible sets are characterized by his consumable income w, and prices $p_1, ..., p_L$ (nonnegative numbers) of the commodities
- Formally, given an income w of the consumer and a price vector $p = (p_1, ..., p_L) \in \mathbb{R}^L_+$, the **budget set** of the consumer is defined by

$$B(p, w) = \left\{ x \in \mathbb{R}^L_+ : \sum_{\ell=1}^L p_\ell x_\ell \leq w \right\}$$

- $B(w, p)$ is a compact ($=$ closed and bounded) and convex set

- The task of finding the $\succsim$ −optimal consumption bundle from $B(p, w)$, i.e. $x^* \in B(p, w)$ such that $x^* \succsim y$ for all $y \in B(p, w)$, is referred as the **consumer's problem**

Remark

If $\succsim$ is nonsatiated, then any optimal consumption bundle x^* in $B(p, w)$ meets Walras' law: $\sum_{\ell=1}^{L} p_\ell x_\ell^* = w$

- Thus any optimal consumption derived from monotonic preferences meets Walras' law too
- But this does not yet guarantee the existence of the optimal choice in a compact set $B(p, w)$ - we know that continuous, convex and monotonic preferences suffice for that (as they are representable by a continuous utility function)
- Are all these conditions required for the existence of optimal bundle?

Proposition

If $\succsim$ is continuous, then there **is** *an optimal consumption bundle in $B(p, w)$*

Proof.

Denote simply $B(p, w) = B$. Suppose that B does not contain a $\succsim -$maximal element. Then for each $y \in B$ there is $x \in B$ such that $x \succ y$. That is $y \in \{z \in B : x \succ z\}$, for some $x \in B$. Then $B \subseteq \cup_{x \in B} \{z \in B : x \succ z\}$. Since preferences are continuous, $\{z \in B : x \succ z\}$ is an open set for all $x \in B$. Since B is a compact set, there is a finite collection $x^1, ..., x^k$ of elements in B such that $B \subseteq \cup_{x \in \{x^1, ..., x^k\}} \{z \in B : x \succ z\}$. But then $B \subseteq \{z \in B : x^* \succ z\}$ for x^* that is $\succsim -$maximal in $\{x^1, ..., x^k\}$. That is, $x^* \succ x$ for all $x \in B$, a contradiction. □

- Thus *only* continuity and rationality of preferences are needed for the existence of an optimal choice; monotonicity not convexity are *not* needed
- Note that the proposition applies to *any* compact set B, not just the budget set $B(p, w)$
- However, the result does not say anything of the properties of the optimal choice - what happens to it if w or p changes?

Proposition

If $\succsim$ is continuous, monotonic, and convex, then the set of optimal consumption bundles in $B(p,w)$ is **convex**

Proof.

Suppose that x and y are optimal, and hence $x \sim y$. Since preferences are convex, $\lambda x + (1-\lambda)y \succsim x$. Since $\sum_\ell p_\ell x_\ell \leq w$ and $\sum_\ell p_\ell y_\ell \leq w$, also $\lambda \sum_\ell p_\ell x + (1-\lambda)\sum_\ell p_\ell y_\ell \leq w$, and hence $\sum_\ell p_\ell[\lambda x_\ell + (1-\lambda)y_\ell] \leq w$. Thus $\lambda x + (1-\lambda)y \in B(p,w)$, implying that also $\lambda x + (1-\lambda)y$ has to be optimal. □

Proposition

If $\succsim$ is continuous, monotonic, and **strictly** *convex, then there is a* **unique** *optimal consumption bundle in $B(w,p)$*

- Denote by $x(w,p)$ the optimal consumption bundle in $B(w,p)$, referred as **Marshallian demand** under w, and p
- Further, $x(\cdot,\cdot)$ is the **Marshallian demand function**, specifying the optimal consumption bundle for each w and p
- An important property of the demand function is that it is not sensitive to small changes in the underlying environment, i.e. it is **continuous** in p and w

Theorem

Let $\succsim$ is continuous, monotonic, and strictly convex and $x(p,w)$ optimal consumption bundle under p,w. Then the Marshallian demand function $x(\cdot,\cdot)$ is continuous in p and w whenever $p_\ell > 0$ for all ℓ and $w > 0$.

- Since under the assumed properties the preferences entertain a continuous utility function, the proof of the theorem follows as a direct corollary of Berge's Maximum Theorem
- However, we will give a direct proof

Proof.

Suppose that x is *not* continuous in p. Then there is p^k converging to p^* such that $x(p^k, w)$ converges to $y^* \neq x^* = x(p^*, w)$ (why?). Since, by continuity, $y^* \in B(p^*, w)$, and since x^* is the optimal choice in $B(p^*, w)$, $x^* \succ y^*$. Thus x^* and y^* have open neighborhoods B_{x^*} and B_{y^*} such that $x \succ y$ for all x and y in these neighborhoods, respectively. Choose $z \in B_{x^*}$ such that $\Sigma_\ell p_\ell z_\ell < w$. For sufficiently high k, also $\Sigma_\ell p_\ell^k z_\ell < w$. But for sufficiently high k, also $x(p^k, w) \in B_{y^*}$ and hence $z \succ x(p^k, w)$, contradicting the assumption that $x(p^k, w)$ is the optimal choice in $B(p^k, w)$. □